Computing Nearest Singularities and Solutions of Polynomial Systems

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Outline

1. Problem Statement
   - computing Taylor series
   - the need for multiple double precision
   - a triangular block Toeplitz system
2. Computing Nearest Singularities
   - logarithmic convergence to a singular solution
   - reconditioning the homotopy
   - software and references

polynomial homotopy continuation

A polynomial homotopy is a system of polynomials in one parameter t, the solution trajectories are then also analytic functions in t, therefore *apply analytic continuation* to approximate the solutions.

Nearby singularities are problematic for convergence.

Theorem (the ratio theorem, Fabry 1896)

If for the series $x(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^n + c_{n+1} t^{n+1} + \cdots$*,*
we have $\lim_{n \rightarrow \infty} c_n / c_{n+1} = z$*, then*

- z *is a singular point of the series, and*
- *it lies on the boundary of the circle of convergence of the series.*

Then the radius of this circle equals $|z|$*.*

The ratio c_n / c_{n+1} is the pole of Padé approximants of degrees $[n/1]$ (n is the degree of the numerator, with linear denominator).

the need for multiple double precision

A *multiple double* is an unevaluated sum of nonoverlapping doubles.

$$\exp(t) = \sum_{k=0}^{d-1} \frac{t^k}{k!} + O(t^d).$$

Assuming the quadratic convergence of Newton's method:

k	$1/k!$	recommended precision	`eps`
7	`2.0e-004`	double precision okay	`2.2e-16`
15	`7.7e-013`	use double doubles	`4.9e-32`
23	`3.9e-023`	use double doubles	
31	`1.2e-034`	use quad doubles	`6.1e-64`
47	`3.9e-060`	use octo doubles	`4.6e-128`
63	`5.0e-088`	use octo doubles	
95	`9.7e-149`	need hexa doubles	`5.3e-256`
127	`3.3e-214`	need hexa doubles	

Software: QDlib (Y. Hida, X. S. Li, and D. H. Bailey),
CAMPARY (M. Joldes, J.-M. Muller, V. Popescu, and W. Tucker).

a triangular block Toeplitz system

Newton's method on power series requires the solution of

$$A(t)\mathbf{x}(t) = \mathbf{b}(t)$$

which, truncated after the 4-th term, *linearizes* into

$$\begin{aligned} A(t) &= A_0 + A_1 t + A_2 t^2 + A_3 t^3, \\ \mathbf{x}(t) &= \mathbf{x}_0 + \mathbf{x}_1 t + \mathbf{x}_2 t^2 + \mathbf{x}_3 t^3, \quad \mathbf{b}(t) = \mathbf{b}_0 + \mathbf{b}_1 t + \mathbf{b}_2 t^2 + \mathbf{b}_3 t^3, \end{aligned}$$

leading to a triangular block Toeplitz system:

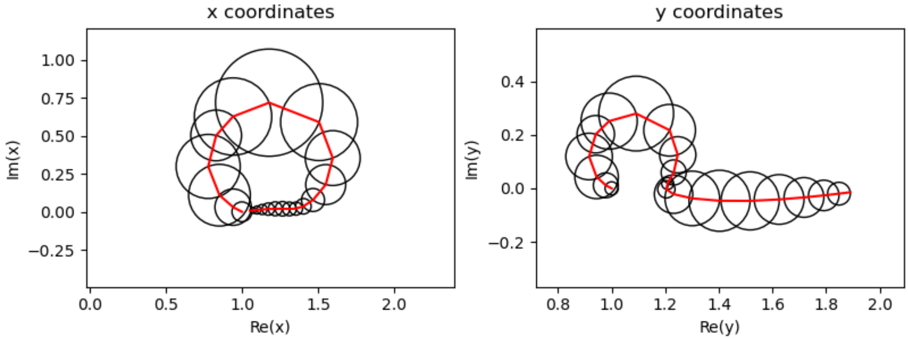
$$\begin{bmatrix} A_0 & & & \\ A_1 & A_0 & & \\ A_2 & A_1 & A_0 & \\ A_3 & A_2 & A_1 & A_0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_0 \\ \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{bmatrix}.$$

logarithmic convergence to a singular solution

The homotopy, for a random complex constant γ,

$$\gamma(1-t)\left(\begin{array}{rcl} x^2 - 1 & = & 0 \\ y^2 - 1 & = & 0 \end{array}\right) + t \left(\begin{array}{rcl} x^2 + y - 3 & = & 0 \\ x + 0.125y^2 - 1.5 & = & 0 \end{array}\right)$$

defines three paths leading to a triple root $(1, 2)$, at $t = 1$.



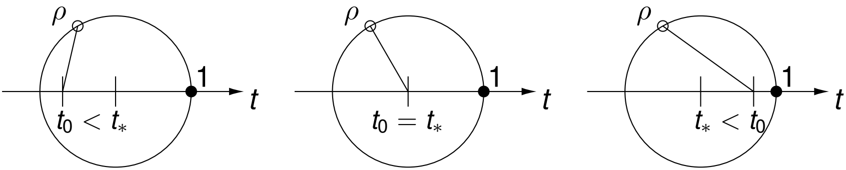
reconditioning the homotopy

At $t = 0.999684$, the application of the theorem of Fabry gives

`(1.0000808949264557-6.886127445687259e-08j)`

as the nearest pole for the example on the previous slide.

The ρ is the last pole before $t = 1$ is detected.



For reliable numerical computations, the formulas

$$t = rs + t_0, \quad r = 1 - t_0, \quad t_0 = t_* + \delta,$$

define the *reconditioning* of the homotopy.

software and references

PHCpack and phcpy are free and open source, on github.

- JV and K. Viswanathan: Extrapolating Solution Paths of Polynomial Homotopies towards Singularities with PHCpack and phcpy. Accepted by the 8th International Congress on Mathematical Software, 2024.

References on the algorithms:

- N. Bliss and JV: The method of Gauss-Newton to compute power series solutions of polynomial homotopies.
  Linear Algebra and its Applications, 2018.
- S. Telen, M. Van Barel, JV: A robust numerical path tracking algorithm for polynomial homotopy continuation.
  SIAM Journal on Scientific Computing, 2020.
- JV and K. Viswanathan: Locating the closest singularity in a polynomial homotopy. In the *Proceedings of the 24th International Workshop on Computer Algebra in Scientific Computing*, 2022.