

# Tropical Algebraic Geometry in Maple

Jan Verschelde

in collaboration with Danko Adrovic

University of Illinois at Chicago  
Department of Mathematics, Statistics, and Computer Science  
<http://www.math.uic.edu/~jan>  
[jan@math.uic.edu](mailto:jan@math.uic.edu)

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# Executive Summary

- Amoebas in Maple
- When do two polynomials have a common factor?

***tropisms give the germs to grow  
the tentacles of the common amoeba***

- tropical algebraic geometry is a new language for classical ideas like the theorem of Puiseux
- the tropical view leads to a symbolic-numeric algorithm
- Extensions:
  - ① singularities
  - ② space curves

# Outline

## 1 Amoebas

- asymptotics of varieties
- compactifying amoebas lead to Newton polygons

## 2 Tentacles

- at infinity the amoeba gets thinner
- normal fans and tropicalization

## 3 Tropisms

- turning the varieties in a particular direction
- certificates for numerical computations

## 4 Germs

- how the amoeba grows from infinity
- regions of convergence of Puiseux series

# Amoebas

logarithms of varieties (G.M. Bergman 1971)

A **variety** is a solution set of a polynomial system.

Varieties are the main objects in algebraic geometry.

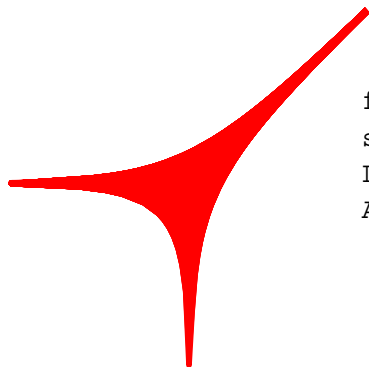
$$\begin{aligned} \log &: \mathbb{C}^* \times \mathbb{C}^* \rightarrow \mathbb{R} \times \mathbb{R} \\ & (x, y) \mapsto (\log(|x|), \log(|y|)) \end{aligned}$$

$$\mathbb{C}^* = \mathbb{C} \setminus \{0\} \quad \text{torus } (\mathbb{C}^*)^2$$

Definition (Gel'fand, Kapranov, and Zelevinsky 1994)

The **amoeba** of a variety is its image under the log map.

# Plotting Amoebas

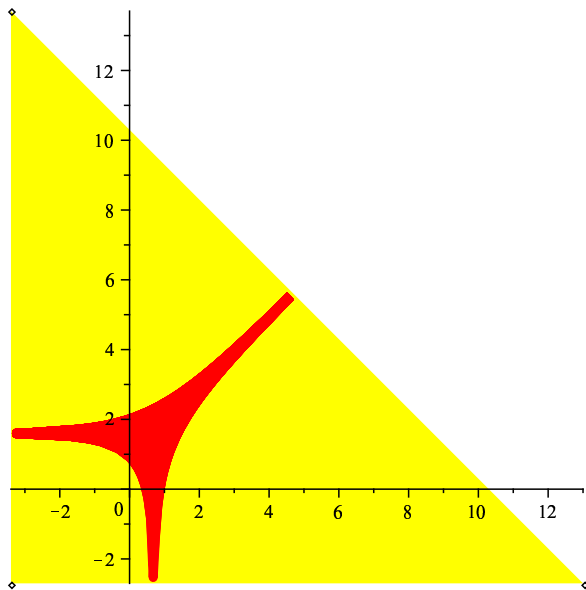


```
f := 1/2*x + 1/5*y - 1:  
s := solve(f,y):  
L := map(log,map(abs,[x,s])):  
A := subs(x=r*exp(I*theta),L);
```

$$A := [\ln(|re^{i\theta}|), \ln(|\frac{5}{2}re^{i\theta} - 5|)]$$

```
> Ap := seq(plot([op(subs(theta=k*Pi/200,A)),  
    r=-100..100],thickness=6),k=0..99):  
> plots[display](Ap,axes=None);
```

# Compactifying the Amoeba



# the Newton polygon

Compactify the amoeba of  $f^{-1}(0)$  as follows:

- 1 take lines perpendicular to the tentacles
- 2 every line cuts the plane in half
- 3 keep half plane where amoeba lives
- 4 intersection of all half planes

→ the Newton polygon of  $f$

## Definition

For  $f(x, y) = \sum_{(i,j) \in A} c_{i,j} x^i y^j$ ,  $c_{i,j} \in \mathbb{C}^*$ .  $A$  is the **support** of  $f$ .

The convex hull of  $A$  is the **Newton polygon**.

The Newton polygon models the sparse structure of a polynomial.

# Detecting Common Factors

problem statement to get some focus

Given are two polynomials  $f$  and  $g$ , in two variables  $x$  and  $y$ , with **approximate** complex coefficients.

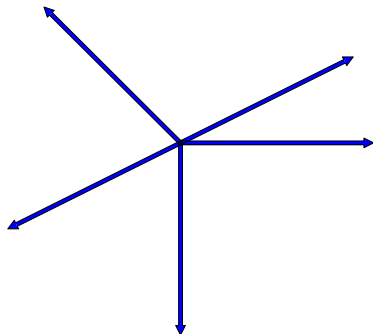
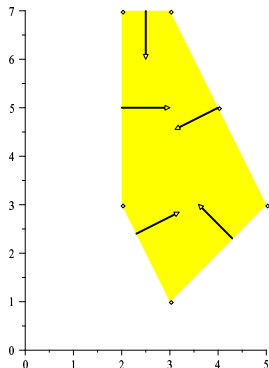
Question: Is there a common factor?

Look at it from a tropical point of view:

- look at the amoeba of the common factor
- the tentacles of the amoeba stretch to infinity
- lines at infinity are edges of the Newton polygon
- look at edges in common direction: **tropisms**

# Inner Normals represent Tentacles

$$f := x^3y + x^2y^3 + x^5y^3 + x^4y^5 + x^2y^7 + x^3y^7$$



The collection of inner normals to the edges of the Newton polygon forms **a tropicalization** of  $f$ .

# Duality links Exponents and Directions

via inner product

Exponents and direction vectors are related through duality.

## Definition

The **inner product** is

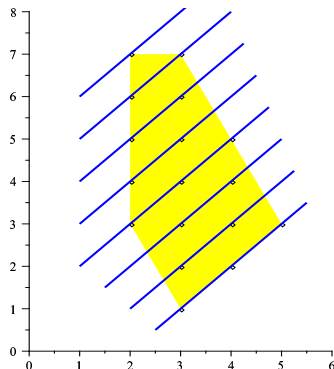
$$\begin{aligned} \langle \cdot, \cdot \rangle : \mathbb{Z}^2 \times \mathbb{Z}^2 &\rightarrow \mathbb{Z} \\ ((i, j), (u, v)) &\mapsto iu + jv. \end{aligned}$$

Given a vector  $(u, v)$ ,  $\langle \cdot, (u, v) \rangle$  ranks the points  $(i, j)$ .

For  $(u, v) = (1, 1)$ , we have the usual degree of  $x^i y^j$ .

## Grading the Points in the Support

Looking at the support in the direction  $(-1, +1)$ :



$$-1 \times i + (+1) \times j = +5$$

$$-1 \times i + (+1) \times j = +4$$

$$-1 \times i + (+1) \times j = +3$$

$$-1 \times i + (+1) \times j = +2$$

$$-1 \times i + (+1) \times j = +1$$

$$-1 \times i + (+1) \times j = 0$$

$$-1 \times i + (+1) \times j = -1$$

$$-1 \times i + (+1) \times j = -2$$

The degree of  $x^i y^j$  in the direction  $(u, v)$  is  $\langle (i, j), (u, v) \rangle$ .

```
Groebner[WeightedDegree](f, [-1, +1], [x, y]);
```

# Normal Fan

a tropicalization of a polynomial

## Definition

Let  $P$  be the Newton polygon of  $f$ .

The **normal cone to a vertex  $p$  of  $P$**  is

$$\{ v \neq 0 \mid \langle p, v \rangle = \min_{q \in P} \langle q, v \rangle \}.$$

The **normal cone to an edge spanned by  $p_1$  and  $p_2$**  is

$$\{ v \neq 0 \mid \langle p_1, v \rangle = \langle p_2, v \rangle = \min_{q \in P} \langle q, v \rangle \}.$$

The **normal fan** of  $P$  is the collection of all normal cones to vertices and edges of  $P$ .

All normal cones to the edges of  $P$  define **a tropicalization of  $f$** .

# Preprocessing Criterium

remember the original question

***Do two polynomials have a common factor?***

We compute tropicalizations of the polynomials.

Recall the tropical view:

- consider the amoeba of the common factor
- the common amoeba must have common tentacle

Amoebas have no common tentacle

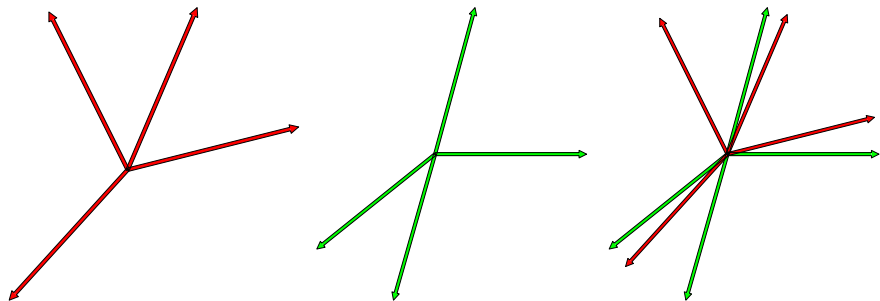
⇒ no common factor.

An efficient preprocessing step independent of the coefficients.

# No Common Factor

implied by polygons in general position

Tropicalized two random polynomials of degree 15

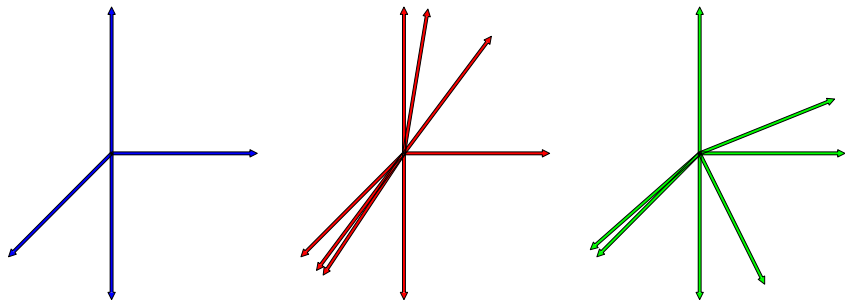


For nonzero coefficients, there can be no common factor.  
Consequence of Bernshtein's second theorem (1975).

# There is a Common Factor

Generated a factor of degree 5  
and multiplied with two random polynomials of degree 10.

Tropicalization of the factor and the two polynomials:



Observe the common tentacles in the tropicalizations.

# Tropisms

for Puiseux expansions (J. Maurer 1980)

screensaver dictionary definition:

*the turning of all or part of an organism in a particular direction in response to an external stimulus*

## Definition

Let  $P$  and  $Q$  be Newton polygons of  $f$  and  $g$ . A **tropism** is a vector perpendicular to one edge of  $P$  and one edge of  $Q$ .

This definition is adapted from Joseph Maurer: *Puiseux expansion for space curves*. Manuscripta Math. 32:91-100, 1980.

# Initial Forms

## **Why are tropisms so important?**

Selecting those monomials which span the edges picked out by the tropism defines a polynomial system which admit a solution in  $(\mathbb{C}^*)^2$ .

### Definition

Let  $(u, v)$  be a direction vector. Consider  $f = \sum_{(i,j) \in A} c_{i,j} x^i y^j$ .

The **initial form of  $f$  in the direction  $(u, v)$**  is

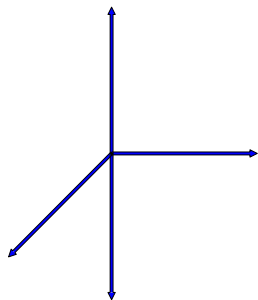
$$\text{in}_{(u,v)}(f) = \sum_{\substack{(i,j) \in A \\ \langle (i,j), (u,v) \rangle = m}} c_{i,j} x^i y^j,$$

where  $m = \min\{ \langle (i,j), (u,v) \rangle \mid (i,j) \in A \}$ .

# Where Tentacles meet Infinity

The factor common to  $f$  and  $g$  is

$$r = 2xy + x^2y + 9xy^2 + 7x^3y + x^4y + 9x^3y^2,$$



Investigate 4 directions, take  $(1, 0)$ :

$$\text{in}_{(1,0)}(r) = 2xy + 9xy^2$$

Initial forms of  $f$  and  $g$ :

$$\text{in}_{(1,0)}(f) = 55xy^6 + 10xy^5 + 45xy^7$$

$$\text{in}_{(1,0)}(g) = 10xy^6 + 45xy^7$$

$\text{in}_{(1,0)}(f) = 5y^5(y+1)(2+9y)/x$  and  $\text{in}_{(1,0)}(g) = 5y^5(2+9y)/x$   
 $\Rightarrow y = -2/9$  represents common root at infinity

# Unimodular Transformations

Investigating the direction  $(-1, -1)$ :

$$\begin{cases} \text{in}_{(-1,-1)}(f) &= 54x^{13}y^2 + 6x^{14}y \\ \text{in}_{(-1,-1)}(g) &= 72x^9y^{10} + 8x^{10}y^9 \end{cases}$$

Change coordinates using  $M = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}$ , unimodular.

Substitute  $x = X^{-1}Y^{-1}$ ,  $y = Y^{-1}$ :

$$\begin{cases} \text{in}_{(-1,-1)}(f) &= (54X + 6)/(X^{14}Y^{15}) \\ \text{in}_{(-1,-1)}(g) &= (72X + 8)/(X^{10}Y^{19}) \end{cases}$$

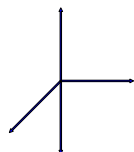
$\Rightarrow X = -1/9$  represents common root at infinity

# Degrees of the Tentacles

$$r = 2xy + x^2y + 9xy^2 + 7x^3y + x^4y + 9x^3y^2$$

The amoeba for  $r$  has four tentacles. A tropicalization is

$$\{ (1, 0), (0, 1), (-1, -1), (0, -1) \}.$$



| $(u, v)$   | $\text{in}_{(u,v)}(r)$      | degree |
|------------|-----------------------------|--------|
| $(1, 0)$   | $2xy + 9xy^2$               | 1      |
| $(0, 1)$   | $2xy + x^2y + 7x^3y + x^4y$ | 3      |
| $(-1, -1)$ | $x^4y + 9x^3y^2$            | 1      |
| $(0, -1)$  | $9xy^2 + 9x^3y^2$           | 2      |

Count the number of nonzero solutions of the initial forms, after proper unimodular coordinate transformation.

# Computing Tropicalizations and Tropisms

cost analysis

The cost of the first preprocessing stage:

- 1  $O(n \log(n))$  operations for computing a tropicalization  $\text{Trop}(f)$  of a polynomial  $f$  with  $n$  monomials ( $n = \#\text{support}(f)$ ).
- 2 Tropisms for  $f$  and  $g$  are normals to edges.

$$\in \text{Trop}(f) \cap \text{Trop}(g).$$

$$\text{cost : } \begin{array}{ccc} O(n \log(n)) & + & O(m \log(m)) \\ n = \#\text{support}(f) & & m = \#\text{support}(g) \end{array}$$

Merge sorted lists of normals in  $\text{Trop}(f)$  and  $\text{Trop}(g)$ .

No tropisms  $\Rightarrow$  no common factor.

# Computing Solutions at Infinity

we have tropisms

The cost of the second preprocessing stage:

For every tropism  $t \in \text{Trop}(f) \cap \text{Trop}(g)$ :

$$\text{solve } \begin{cases} \text{in}_t(f)(z) = 0 \\ \text{in}_t(g)(z) = 0 \end{cases} \quad z \in (\mathbb{C}^*)^2.$$

Perform unimodular transformation:  $t$  becomes  $(1, 0)$ .

Cost per tropism:

$O((n + m)^3)$  via SVD (worst case)

$n = \#\text{support}(f)$  and  $m = \#\text{support}(g)$

# Growing Tentacles from Infinity

Once we have  $z \in (\mathbb{C}^*)^2$ :  $\text{in}_t(f)(z) = 0 = \text{in}_t(g)(z)$ ,

there are two more steps:

- 1 Is the solution at infinity isolated ?
- 2 How to represent the common factor?

→ apply the Newton-Puiseux method

extend proof in Robert Walker's *algebraic curves*  
towards Joseph Maurer's general method

# Fractional Power Series

or Puiseux series

The canonical form of a power series of a curve.

## Definition

Consider the curve defined by  $f(x, y) = 0$ . A **fractional power series** has the form

$$\begin{cases} x &= z_1 t^u \\ y &= z_2 t^v (1 + O(t)) \end{cases} \quad (z_1, z_2) \in (\mathbb{C}^*)^2.$$

The leading exponents  $(u, v)$  are the tropisms.

The leading coefficients  $(z_1, z_2)$  satisfy

$$\begin{cases} \text{in}_{(u,v)}(f)(z_1, z_2) = 0 \\ \text{in}_{(u,v)}(g)(z_1, z_2) = 0 \end{cases}$$

# Extending Solution at Infinity

example continued

Tropism  $e = (1, 0)$  hints at a common factor.

```
sort({ f , g }, plex, ascending)
```

$$\begin{cases} f = 10xy^5 + 45xy^7 + 55xy^6 + x^2(30 \text{ other terms}) \\ g = 45xy^7 + 10xy^6 + x^2(34 \text{ other terms}) \end{cases}$$

Let  $f_1 = f/x$  and  $g_1 = g/x$ , then  $z = (0, -2/9)$  is solution at infinity.

$$\begin{cases} x = t^1 \\ y = -\frac{2}{9}t^0 + ct(1 + O(t)), \quad c \in \mathbb{C}^*. \end{cases}$$

A nonzero value for  $c$  shows existence of common factor.

# Computing the Power Series

```
zt := x = t, y = -2/9 + C*t;  
f1z := subs(zt,f1): g1z := subs(zt,g1):  
c1 := coeff(f1z,t,1); c2 := coeff(g1z,t,1);
```

$$c1 = -\frac{1120}{531441} - \frac{1120}{59049} C$$

$$c2 = -\frac{320}{59049} - \frac{320}{531441} C$$

Solving both equations for  $C$  gives  $C = -1/9$ .

$$\begin{cases} x = t \\ y = -\frac{2}{9} - \frac{1}{9}t(1 + O(t)). \end{cases}$$

Substituting  $x = t, y = -2/9 - t/9$  into  $f_1$  and  $g_1$  gives  $O(t^2)$ .

# Germ

neighborhoods where Puiseux series converge

Towards a polyhedral method for curves...

Expanding the power series to a suitable order, we sample points from the curve, starting at infinity after a proper coordinate change.

## Definition

Given a power series expansion at a point, a **germ** is an equivalence class of open neighborhoods of the point where the series converge.

From *Local Analytic Geometry* by Theo de Jong and Gerhard Pfister.

Numerical homotopy continuation algorithms for finding representations for curves by moving the hyperplanes at infinity to hyperplane sections in affine space.

# Conclusions and Extensions

Like Maple, tropical algebraic geometry is language.

Sentences like

***tropisms give the germs to grow  
the tentacles of the common amoeba***

lead to efficient preprocessing stages to detect and compute common factors of two polynomials with approximate coefficients.

Extensions:

- 1 handle singularities numerically,
- 2 develop a polyhedral methods for curves.

Further reading: *Solving Systems of Polynomial Equations*  
by Bernd Sturmfels, AMS 2002.