0. The Problem
Solve \( f(\lambda, x) = 0 \)
for the variables \( x \), where \( \lambda \) are parameters;
\( f \) is a polynomial system with approximate coefficients.
Solve means: by computer using general methods
\( \rightarrow \) so any one can re-solve and verify.
Three categories of solutions:
(1) approximations to all isolated solutions;
(2) all irreducible solution components, of all dimensions;
(3) information about the exceptional parameter values.

1. Parameter Continuation

A generic choice for start avoids singularities along the paths.

2. Polyhedral Homotopies
Newton polytopes model sparse structure:
\[
\begin{align*}
F_1(x) &= a_{1,2}x_2^2 + a_{1,3}x_2 + a_{1,4}x_4 \\
F_2(x) &= b_{1,2}x_2^2 + b_{1,3}x_2 + b_{1,4}x_4 + b_{1,5}
\end{align*}
\]
Bernstein: mixed volume bounds isolated roots in \( \mathbb{C}^d \).
Polyhedral homotopies are optimal for generic coefficients.

References

3. Numerical Irreducible Decomposition
Numerical representations of positive dimensional solution sets.

Cut space curve with a random plane to find its degree.

4. Monodromy Factorization
The Riemann Surface of \( z^3 - w = 0 \):

Loop around the singular point (0,0) permutes the points.

5. Deflation for Isolated Singularities
Jacobian matrix \( J_f(x) \) rank deficient close to \( x^* \);
let \( R \) be the numerical rank of \( J_f(x^*) \); and
introduce \( R+1 \) extra multiplier variables \( \mu \).

Apply Newton to
\[
\begin{align*}
\{ & f(x) = 0 \quad \mu \quad \text{is the multiplier vector} \\
& J_f(x)B \mu = 0 \\
& (c, \mu) \quad \text{is a random matrix} \\
& (e, \mu) \quad \text{is a random vector}
\end{align*}
\]

Reduced to corank one case. Repeat if necessary.

References

6. Applications to Mechanisms
Chebyshev’s 4-bar mechanism and its cognates:

Given points the mechanism must reach, determine its parameters.

7. Applications to Control
Control of an \( m \)-input and \( p \)-output plant by a \( q \)th order dynamic compensator:

\[
\begin{align*}
& u \in \mathbb{R}^m & x \in \mathbb{R}^d & y \in \mathbb{R}^d \\
& x = Ax + Bu \\
& y = Cx \\
& \dot{z} = Fz + Gy \\
& u = Hz + Ky
\end{align*}
\]

Every feedback law corresponds to a polynomial map of degree \( q \) into the Grassmannian of \( m \)-planes in \( \mathbb{C}^{m+p} \) that meet \( mp + q(m+p) \) given \( m \)-planes sampled at \( mp + q(m+p) \) interpolation points.

8. Software on Clusters and Supercomputers
PHCpack [2] is available in source form, with binaries for many different computers. Interfaces: PHCmaple (for Maple), PHClab (for MATLAB and Octave).
PHClib offers C wrappers to treat the code as a library. The parallel path trackers of PHCpack use PHClib and MPI, developed on Rocketcale personal clusters. PHCpack is among the experimental packages in SAGE.

References

Acknowledgements
This material is based upon work supported by the National Science Foundation under Grants No. 0105739, 0134611, 0410036, and 0713018, from DMS - Computational Mathematics (with contributions from ANTC). Thanks to NCSA for access to Copper and Tungsten via TG-CCR060019N: Parallel Numerical Algebraic Geometry.