

# Polyhedral Methods for Space Curves Exploiting Symmetry\*

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## Abstract

We present a polyhedral algorithm to manipulate positive dimensional solution sets. Using facet normals to Newton polytopes as pretropisms, we focus on the first two terms of a Puiseux series expansion. The leading powers of the series are computed via the tropical prevariety. This polyhedral algorithm is well suited for exploitation of symmetry, when it arises in systems of polynomials. Initial form systems with pretropisms in the same group orbit are solved only once, allowing for a systematic filtration of redundant data. Computations with `cddlib` and `Sage` are illustrated on cyclic  $n$ -roots polynomial systems.

## 1 Introduction

We consider a polynomial system  $f(\mathbf{x}) = \mathbf{0}$ ,  $\mathbf{x} = (x_1, x_2, \dots, x_n)$ ,  $f = (f_1, f_2, \dots, f_n)$ ,  $f_i \in \mathbb{C}[\mathbf{x}]$ ,  $i = 1, 2, \dots, n$ . While in many applications, the coefficients of the polynomials are rational numbers, we allow the input system to have approximate complex numbers as coefficients. As  $f(\mathbf{x}) = \mathbf{0}$  has as many equations as variables, we expect in general to find only isolated solutions. In this paper we focus on systems with particular choices of the coefficients so  $f(\mathbf{x}) = \mathbf{0}$  has a space curve as solution.

Our approach is based on the following observation: if the solution set of  $f(\mathbf{x}) = \mathbf{0}$  has a space curve, then this space curve extends from  $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$  to infinity. In particular: the space curve intersects hyperplanes at infinity in isolated points. We start our series development of the space curve at these isolated points. We applied this approach to find common factors of two polynomials in [1] and a first attempt to generalize this approach to any dimension occurred in [23]. Our approach can be interpreted as a symbolic-numeric version of the effective proof of the fundamental theorem of tropical algebraic geometry in [12].

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The hyperplanes at infinity are associated to the facets of the Newton polytopes of the polynomial system. In [10], Newton polytopes are introduced as compactifications of logarithms of algebraic sets (also known as amoebas). Formally we denote a polynomial  $p \in \mathbb{C}[\mathbf{x}]$  as

$$p(\mathbf{x}) = \sum_{\mathbf{a} \in A} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}, \quad c_{\mathbf{a}} \in \mathbb{C}^*, \quad \mathbf{x}^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}, \quad (1)$$

and we call the set  $A$  of exponents the support of  $p$ . The convex hull of  $A$  is then the Newton polytope of  $p$ . To every facet of  $P$  corresponds a hyperplane at infinity and for every facet we can then introduce projective coordinates [22].

In this paper we will make various significant assumptions. First we assume that the space curves we consider are reduced, that is: free of multiplicities. Moreover, the space curve is in general position with respect to the first coordinate plane. More precisely: we assume that the space curve is not contained in the coordinate plane defined by  $x_1 = 0$ . Thirdly, we assume the space curve intersects the plane  $x_1 = 0$  at regular solutions. The special role of  $x_1$  is reflected in the normal form of Puiseux series:

$$\begin{cases} x_1 = t^{v_1} \\ x_i = t^{v_i} (y_i + z_i t^{w_i}), \quad i = 2, \dots, n. \end{cases} \quad (2)$$

where the leading powers  $\mathbf{v} = (v_1, v_2, \dots, v_n)$  equal to what is called a tropism. The leading coefficients  $y_i \in \mathbb{C}^*$  are defined as solutions of the initial form system defined by the tropism  $\mathbf{v}$ .

Our approach consists of two stages:

1. The computation of the pretropisms, which are candidates for the leading powers of the Puiseux series [14], [15], [25]. We describe this computation in section 4.
2. The computation of the leading coefficient and the second term of the Puiseux series, described in the fifth section.

The second term of the Puiseux series indicates the existence of a space curve. The focus of our approach is on systems that have (at least) as many equations as variables, i.e.: for system where one would normally not expect to have space curves as solutions.

An obvious advantage of our approach is the exploitation of symmetry. If the system is invariant to permutation of the variables, then it suffices to compute only the generators of the solution orbits. We then develop the Puiseux series only at the generators. As we illustrate first on the cyclic 5-roots system, this approach of exploiting symmetry is not limited to systems with space curves as solutions.

Our running example is one family of polynomial systems. While still preliminary and limited in scope, we are encouraged by the results on the cyclic 12-roots problem. In particular, we obtain exact representations for the solution curves derived from numerical homotopy solvers. Our results on cyclic 12-roots correspond to [16].

## 2 the Cyclic $n$ -roots Problem

Our running example throughout this paper is the cyclic  $n$ -roots problem, an academic benchmark for polynomial system solvers, see e.g: [4], [6], [8], [13]. For  $n = 3$ , the system originates naturally

from the elementary symmetric functions in the roots of a cubic polynomial. For  $n = 4$ , the system is more interesting:

$$f(\mathbf{x}) = \begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1 = 0 \\ x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_1 + x_4x_1x_2 = 0 \\ x_1x_2x_3x_4 - 1 = 0 \end{cases} \quad (3)$$

We illustrate the application of our approach to this system. Terminology and precise definitions are in later sections. There is only one tropism  $\mathbf{v} = (+1, -1, +1, -1)$  defining the initial form  $\text{in}_{\mathbf{v}}(f)(\mathbf{z}) = \mathbf{0}$ :

$$\text{in}_{\mathbf{v}}(f)(\mathbf{x}) = \begin{cases} x_2 + x_4 = 0 \\ x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1 = 0 \\ x_2x_3x_4 + x_4x_1x_2 = 0 \\ x_1x_2x_3x_4 - 1 = 0 \end{cases} \quad \begin{cases} x_1 = y_1^{+1} \\ x_2 = y_1^{-1}y_2 \\ x_3 = y_1^{+1}y_3 \\ x_4 = y_1^{-1}y_4 \end{cases} \quad (4)$$

The system  $\text{in}_{\mathbf{v}}(f)(\mathbf{y}) = \mathbf{0}$  has two solutions. These two solutions are the leading coefficients in the Puiseux series. In this case, the leading term of the series vanishes entirely at the system so we write two solution curves as  $(t, -t^{-1}, -t, t^{-1})$  and  $(t, t^{-1}, -t, -t^{-1})$ . To compute the degree of the two solution curves, we take a random hyperplane in  $\mathbb{C}^4$ :  $c_0 + c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 = 0$ ,  $c_i \in \mathbb{C}^*$ . Then the number of points on the curve and on the random hyperplane equals the degree of the curve. Substituting the representations we obtained for the curves into the random hyperplanes gives a quadratic polynomial in  $t$  (after clearing the denominator  $t^{-1}$ ), so there are two quadric curves of cyclic 4-roots.

### 3 Exploiting Symmetry

We illustrate the exploitation of permutation symmetry on the cyclic 5-roots system. All Newton polytopes of the cyclic 5-roots system are in general position. Therefore, the mixed volume is sharp and its value is 70. We, hence, know that the solution set for the cyclic 5-roots systems consists of 70 isolated solutions. Adjusting polyhedral homotopies to exploit the permutation symmetry for this system was presented in [24]. Here we present an alternative approach.

If we consider the first four equations of the cyclic 5-roots system  $C_5(\mathbf{x}) = \mathbf{0}$ , then we have solution curves. Consider the first four equations of  $C_5(\mathbf{x}) = \mathbf{0}$ :

$$\begin{cases} x_0 + x_1 + x_2 + x_3 + x_4 = 0 \\ x_0x_1 + x_0x_4 + x_1x_2 + x_2x_3 + x_3x_4 = 0 \\ x_0x_1x_2 + x_0x_1x_4 + x_0x_3x_4 + x_1x_2x_3 + x_2x_3x_4 = 0 \\ x_0x_1x_2x_3 + x_0x_1x_2x_4 + x_0x_1x_3x_4 + x_0x_2x_3x_4 + x_1x_2x_3x_4 = 0 \end{cases} \quad (5)$$

where  $\mathbf{v} = (1, 1, 1, 1, 1)$ . As the first four equations of  $C_5$  are homogeneous, the first four equations of  $C_5$  coincide with the first four equations of  $\text{in}_{\mathbf{v}}(C_5)(\mathbf{x}) = \mathbf{0}$ . Because these four equations are homogeneous, we have lines of solutions. After computing representations for the solution lines, we find the solutions to the original cyclic 5-roots problem intersecting the solution lines with the

hypersurface defined by the last equation. In this intersection, the exploitation of the symmetry is straightforward.

The unimodular matrix with  $\mathbf{v} = (1, 1, 1, 1, 1)$  is given by

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (6)$$

The corresponding unimodular coordinate transformation, denoted as  $\mathbf{x} = \mathbf{z}^M$ , is defined as

$$x_0 = z_0, \quad x_1 = z_0 z_1, \quad x_2 = z_0 z_2, \quad x_3 = z_0 z_3, \quad x_4 = z_0 z_4 \quad (7)$$

Applying  $\mathbf{x} = \mathbf{z}^M$  to the initial form system (5) gives

$$\text{in}_{\mathbf{v}}(C_8)(\mathbf{x} = \mathbf{z}^M) = \begin{cases} z_1 + z_2 + z_3 + z_4 + 1 = 0 \\ z_1 z_2 + z_2 z_3 + z_3 z_4 + z_1 + z_4 = 0 \\ z_1 z_2 z_3 + z_2 z_3 z_4 + z_1 z_2 + z_1 z_4 + z_3 z_4 = 0 \\ z_1 z_2 z_3 z_4 + z_1 z_2 z_3 + z_1 z_2 z_4 + z_1 z_3 z_4 + z_2 z_3 z_4 = 0 \end{cases} \quad (8)$$

The system (8) has 14 isolated solutions of the form  $z_1 = c_1, z_2 = c_2, z_3 = c_3, z_4 = c_4$ . If we let  $z_0 = t$ , in the original coordinates we have

$$x_0 = t, \quad x_1 = t c_1, \quad x_2 = t c_2, \quad x_3 = t c_3, \quad x_4 = t c_4 \quad (9)$$

as representations for the 14 solution lines.

Substituting (9) into the omitted equation  $x_0 x_1 x_2 x_3 x_4 - 1 = 0$ , yields a univariate polynomial in  $t$  of the form  $k t^5 - 1 = 0$ , where  $k$  is a constant.

Among the 14 solutions, 10 are of the form  $t^5 - 1$ . They account for  $10 \times 5 = 50$  solutions. There are two solutions of the form  $(-122.99186938124345)t^5 - 1$ , accounting for  $2 \times 5 = 10$  solutions and an additional two solutions are of the form  $(-0.0081306187557833118)t^5 - 1$  accounting for  $2 \times 5 = 10$  remaining solutions. The total number of solutions is 70, as indicated by the mixed volume computation.

Existence of additional symmetry, which can be exploited, can be seen in the relationship between the coefficients of the quintic polynomial, i.e.  $\frac{1}{(-122.99186938124345)} \approx -0.0081306187557833118$ .

The next cases in the family that admit positive dimensional solution sets are for  $n = 8$  and  $n = 9$  as both 8 and 9 has a square divisor [2]. With more efficient implementations of the approach described in this paper, also the cyclic 12-roots problem comes within reach.

That the first  $n - 1$  equations of cyclic  $n$ -roots system give explicit solution lines is exceptional. In general, we plan to use the leading term of the Puiseux series to compute witness sets for the space curves defined by the first  $n - 1$  equations. Then via the diagonal homotopy [17] we can intersect the space curves with the rest of the system. While the direct exploitation of symmetry with witness sets is not possible, with the Puiseux series we can pick out the generating space curves.

## 4 Computing Pretropisms

Let  $\mathbf{v} \neq \mathbf{0}$  and denote  $\langle \mathbf{a}, \mathbf{v} \rangle = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$ . Then  $\text{in}_{\mathbf{v}}(f)$  is the initial form of  $f$  in the direction of  $\mathbf{v}$ :

$$\text{in}_{\mathbf{v}}(f) = \sum_{\substack{\mathbf{a} \in A \\ \langle \mathbf{a}, \mathbf{v} \rangle = m}} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}, \quad \text{for } f = \sum_{\mathbf{a} \in A} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}} \quad (10)$$

where  $m = \min\{ \langle \mathbf{a}, \mathbf{v} \rangle \mid \mathbf{a} \in A \}$ . For the connections with Gröbner bases, see [20]. For  $f$  supported by  $A$ , denote the support of  $\text{in}_{\mathbf{v}}(f)$  by  $\text{in}_{\mathbf{v}}(A)$ . A system  $f = (f_1, f_2, \dots, f_n)$  is supported on  $(A_1, A_2, \dots, A_n)$ .

We look for  $\mathbf{v}$  so that  $\text{in}_{\mathbf{v}}(f)(\mathbf{x}) = \mathbf{0}$  has solutions in  $(\mathbb{C}^*)^n$ . A nonzero vector  $\mathbf{v}$  is a *pretropism* for the system  $f(\mathbf{x}) = \mathbf{0}$  if  $\#\text{in}_{\mathbf{v}}(f_k) \geq 2$  for all  $k$  ranging from 1 to  $n$ . Every tropism is a pretropism, but not every pretropism is a tropism, as pretropisms depend only on supports  $A = (A_1, A_2, \dots, A_n)$  of  $f$ .

The Cayley trick formulates a resultant as a discriminant as in [10, Proposition 1.7, page 274]. Here we follow the geometric description of [19], see also [7, §9.2]. Via the Cayley embedding we reduce  $A$  to *one* set:

$$E_A = (A_1 \times \{\mathbf{0}\}) \cup (A_2 \times \{\mathbf{e}_1\}) \cup \cdots \cup (A_n \times \{\mathbf{e}_{n-1}\}) \quad (11)$$

where  $\mathbf{e}_k$  is the  $k$ -th  $(n-1)$ -dimensional unit vector. We call the convex hull of the Cayley embedding the Cayley polytope  $\text{conv}(E_A)$

We claim that enumerating all facet normals to  $\text{conv}(E_A)$  yields all tropisms. Furthermore: tropisms for curves are normals to facets spanned by at least two points of each support. In case of surfaces of solutions, we will find cones spanned by tropisms for curves.

Algorithms to compute tropical varieties are described in [5] and implemented in Gfan [11].

Following from the second theorem of Bernshtein [3], the Newton polytopes may be in general position and no normals to at least one edge of every Newton polytope exists. In that case, there does not exist a positive dimensional solution set either.

Running `cddlib` [9] to compute the H-representation of the Cayley polytope of the cyclic 8-roots problem yields 94 pretropisms. With symmetry we have 11 generators, displayed in Table 1.

For the cyclic 9-roots problem, the computation of the facets of the Cayley polytope yield 276 pretropisms, with 17 generators:  $(-2, 1, 1, -2, 1, 1, -2, 1, 1)$ ,  $(-1, -1, 2, -1, -1, 2, -1, -1, 2)$ ,  $(-1, 0, 0, 0, 0, 1, -1, 1, 0)$ ,  $(-1, 0, 0, 0, 0, 1, 0, -1, 1)$ ,  $(-1, 0, 0, 0, 1, -1, 0, 1, 0)$ ,  $(-1, 0, 0, 0, 1, -1, 1, 0, 0)$ ,  $(-1, 0, 0, 0, 1, 0, -1, 0, 1)$ ,  $(-1, 0, 0, 0, 1, 0, -1, 1, 0)$ ,  $(-1, 0, 0, 0, 1, 0, 0, -1, 1)$ ,  $(-1, 0, 0, 1, -1, 0, 1, -1, 1)$ ,  $(-1, 0, 0, 1, -1, 0, 1, 0, 0)$ ,  $(-1, 0, 0, 1, -1, 1, -1, 0, 1)$ ,  $(-1, 0, 0, 1, -1, 1, -1, 1, 0)$ ,  $(-1, 0, 0, 1, -1, 1, 0, -1, 1)$ ,  $(-1, 0, 0, 1, 0, -1, 1, -1, 1)$ ,  $(-1, 0, 0, 1, 0, -1, 1, -1, 1)$ , and  $(-1, 0, 1, -1, 1, -1, 0, 1, 0)$ .

The computations for  $n = 8$  and  $n = 9$  finished in less than a second on one core of a 3.07Ghz Linux computer with 4Gb RAM. For the cyclic 12-roots problem, `cddlib` needed about a week to compute the 907,923 facets normals of the Cayley polytope. Note that only a small fragment

pretropism $\mathbf{v}$	#solutions of $\text{in}_{\mathbf{v}}(C_8)(\mathbf{z})$
$(-3, 1, 1, 1, -3, 1, 1, 1)$	94
$(-1, -1, -1, 3, -1, -1, -1, 3)$	115
$(-1, -1, 1, 1, -1, -1, 1, 1)$	112
$(-1, 0, 0, 0, 1, -1, 1, 0)$	30
$(-1, 0, 0, 0, 1, 0, -1, 1)$	23
$(-1, 0, 0, 1, -1, 1, 0, 0)$	32
$(-1, 0, 0, 1, 0, -1, 1, 0)$	40
$(-1, 0, 0, 1, 0, 0, -1, 1)$	16
$(-1, 0, 1, -1, 1, -1, 1, 0)$	39
$(-1, 0, 1, 0, -1, 1, -1, 1)$	23
$(-1, 1, -1, 1, -1, 1, -1, 1)$	509

Table 1: Eleven pretropism generators of the cyclic 8-root problem, along with the number of solutions of the corresponding initial form systems.

of the total number of facets yield pretropisms, so specific polyhedral algorithms for pretropisms will perform much better.

## 5 The Second Term of a Puiseux Series

In exceptional cases like the cyclic 4-roots problem where the first term of the series gives an exact solution or when we encounter solution lines like with the first four equations of cyclic 5-roots, we do not have to look for a second term of a series. In general, a pretropism  $\mathbf{v}$  becomes a tropism if there is a Puiseux series with leading powers equal to  $\mathbf{v}$ . The leading coefficients of the series is a solution in  $\mathbb{C}^*$  of the initial form system  $\text{in}_{\mathbf{v}}(f)(\mathbf{x}) = \mathbf{0}$ . We solve the initial form systems with PHCpack [21]. For the computations of the series we use Sage [18].

In our approach, the calculation of the second term in the Puiseux series is critical to decide whether a solution of an initial form system corresponds to an isolated solution at infinity of the original system, or whether it constitutes the beginning of a space curve. For sparse systems, we may not assume that the second term of the series is linear in  $t$ . Trying consecutive powers of  $t$  will be wasteful for high degree second terms of particular systems. In this section we explain our algorithm to compute the second term in the Puiseux series.

A unimodular coordinate transformation  $\mathbf{x} = \mathbf{z}^M$  with  $M$  having as first row the vector  $\mathbf{v}$  turns the initial form system  $\text{in}_{\mathbf{v}}(f)(\mathbf{x}) = \mathbf{0}$  into  $\text{in}_{\mathbf{e}_1}(f)(\mathbf{z}) = \mathbf{0}$  where  $\mathbf{e}_1 = (1, 0, \dots, 0)$  equals the first standard basis vector. When  $\mathbf{v}$  has negative components, solutions of  $\text{in}_{\mathbf{v}}(f)(\mathbf{x}) = \mathbf{0}$  that are at infinity (in the ordinary sense of having components equal to  $\infty$ ) are turned into solutions in  $(\mathbb{C}^*)^n$  of  $\text{in}_{\mathbf{e}_1}(f)(\mathbf{z}) = \mathbf{0}$ .

**Proposition 5.1.** *If the initial root does not satisfy the entire transformed polynomial system, then there must be at least one nonzero constant exponent  $a_i$  forming monomial  $c_i t^{a_i}$ .*

*Proof.* Let  $\mathbf{z} = (z_1, z_2, \dots, z_n)$  and  $\bar{\mathbf{z}} = (z_2, z_3, \dots, z_n)$  denote variables after the unimodular

transformation. Let  $(z_1 = t, z_2 = r_2, \dots, z_n = r_n)$  be a regular solution at infinity and  $t$  the free variable.

The  $i$ th equation of the original system after the unimodular coordinate transformation has the form

$$f_i = z_1^{m_i}(P_i(\bar{\mathbf{z}}) + O(z_1)Q_i(\mathbf{z})), \quad i = 1, 2, \dots, n, \quad (12)$$

where the polynomial  $P_i(\bar{\mathbf{z}})$  consists of all monomials which form the initial form component of  $f_i$  and  $Q_i(\mathbf{z})$  is a polynomial consisting of all remaining monomials of  $f_i$ .

For the second term in the series expansions, we denote

$$\begin{aligned} z_1 &= t \\ z_i &= r_i + k_i t^w, \quad i = 2, \dots, n. \end{aligned} \quad (13)$$

We first show that polynomial  $z_1^{m_i} P_i(\bar{\mathbf{z}})$  cannot contain a monomial of the form  $c_i t^{a_i}$  on substitution of (13). The polynomial  $z_1^{m_i} P_i(\bar{\mathbf{z}})$  is the initial form of  $f_i$ , hence solution at infinity  $(z_1 = t, z_2 = r_2, z_3 = r_3, \dots, z_n = r_n)$  satisfies  $z_1^{m_i} P_i(\bar{\mathbf{z}})$  entirely. Substituting (13) into  $z_1^{m_i} P_i(\bar{\mathbf{z}})$  eliminates all constants in  $t^{m_i} P_i(\bar{\mathbf{z}})$ . Hence, the polynomial  $P_i(t) = R_i(t^w)$  and, therefore,  $t^{m_i} P_i(\bar{\mathbf{z}}) = R_i(t^{w+m_i})$ .

We next show that polynomial  $Q_i(\mathbf{z})$  contains a monomial  $c_i t^{a_i}$ . The polynomial  $Q_i(\mathbf{z})$  can be rewritten as

$$z_1^{a_i} Q_i(\bar{\mathbf{z}}) = z_1^{a_i} T_{i0}(\bar{\mathbf{z}}) + z_1^{a_i+1} T_{i1}(\bar{\mathbf{z}}) + \dots + z_1^{a_i+n} T_{in}(\bar{\mathbf{z}}). \quad (14)$$

The polynomial  $Q_i(\mathbf{z}) = z_1^{a_i} Q_i(\bar{\mathbf{z}})$  consists of monomials which are not part of the initial form of  $f_i$ . Hence, on substitution of solution at infinity (13),  $z_1^{a_i} Q_i(\bar{\mathbf{z}}) = t^{a_i} Q_i(t)$  does not vanish entirely and it must contain monomials which are constants. Since  $Q_i(t)$  contains monomials which are constants,  $t^{a_i} Q_i(t)$  must contain a monomial of the form  $c_{ij} t^{a_i}$ .  $\square$

If the initial root does not satisfy the original system, then we may have a second term in the Puiseux series. Assume the following general form of the series:

$$\begin{cases} z_1 = t \\ z_i = c_i^{(0)} + y_i t^w, \quad i = 2, \dots, n, \end{cases} \quad (15)$$

where  $c_i^{(0)} \in \mathbb{C}^*$  are the coordinates of the initial root,  $y_i$  is the unknown coefficient of the second term  $t^w$ ,  $w > 0$ . Note that only for some  $y_i$  nonzero values may exist. We are looking for the smallest  $w$  for which the linear system in the  $y_i$ 's admits a solution with at least one nonzero coordinate. Substituting (15) gives equations of the form

$$\widehat{c_i^{(0)}} t^{a_i} (1 + O(t)) + t^{w+b_i} \sum_{j=1}^n \gamma_{ij} y_j (1 + O(t)) = 0, \quad i = 1, 2, \dots, n, \quad (16)$$

for constant exponents  $a_i$ ,  $b_i$  and constant coefficients  $\widehat{c_i^{(0)}}$  and  $\gamma_{ij}$ .

In the equations of (16) we truncate the  $O(t)$  terms and retain those equations with the smallest value of the exponents  $a_i$ , because with the second term of the series solution we want to eliminate the lowest powers of  $t$  when we plug in the first two terms of the series in the system.

This gives a condition on the value  $w$  of the unknown exponent of  $t$  in the second term. If there is no value for  $w$  so that we can match with  $w + b_i$  the minimal value of  $a_i$  for all equations where the same minimal value of  $a_i$  occurs, then there does not exist a second term and hence no space curve. Otherwise, with the matching value for  $w$  we obtain a linear system in the unknown  $y$  variables. If a solution to this linear system exists with at least one nonzero coordinate, then we have found a second term, otherwise, there is no space curve.

## 6 Series of Cyclic 8 and 12-Roots

### 6.1 cyclic 8-roots

We illustrate our approach on the cyclic 8-roots problem, denoted by  $C_8(\mathbf{x}) = \mathbf{0}$  and take as pretropism  $\mathbf{v} = (1, -1, 0, 1, 0, 0, -1, 0)$ . The corresponding initial form system is

$$\text{in}_{\mathbf{v}}(C_8)(\mathbf{x}) = \begin{cases} x_1 + x_6 = 0 \\ x_1x_2 + x_5x_6 + x_6x_7 = 0 \\ x_4x_5x_6 + x_5x_6x_7 = 0 \\ x_0x_1x_6x_7 + x_4x_5x_6x_7 = 0 \\ x_0x_1x_2x_6x_7 + x_0x_1x_5x_6x_7 = 0 \\ x_0x_1x_2x_5x_6x_7 + x_0x_1x_4x_5x_6x_7 + x_1x_2x_3x_4x_5x_6 = 0 \\ x_0x_1x_2x_4x_5x_6x_7 + x_1x_2x_3x_4x_5x_6x_7 = 0 \\ x_0x_1x_2x_3x_4x_5x_6x_7 - 1 = 0 \end{cases} \quad (17)$$

A unimodular matrix  $M \in \mathbb{Z}^{n \times n}$  is an invertible integer matrix. We use unimodular matrices for coordinate transformations. Our unimodular matrices have the tropism as their first row:

$$M = \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \quad (18)$$

Then the corresponding unimodular coordinate transformation, denoted as  $\mathbf{x} = \mathbf{z}^M$ , is defined as

$$x_0 = z_0, \quad x_1 = z_1/z_0, \quad x_2 = z_2, \quad x_3 = z_0z_3, \quad x_4 = z_4, \quad x_5 = z_5, \quad x_6 = z_6/z_0, \quad x_7 = z_7 \quad (19)$$

Applying  $\mathbf{x} = \mathbf{z}^M$  to the initial form system (17) gives

$$\text{in}_{\mathbf{v}}(C_8)(\mathbf{x} = \mathbf{z}^M) = \begin{cases} z_1 + z_6 = 0 \\ z_1 z_2 + z_5 z_6 + z_6 z_7 = 0 \\ z_4 z_5 z_6 + z_5 z_6 z_7 = 0 \\ z_4 z_5 z_6 z_7 + z_1 z_6 z_7 = 0 \\ z_1 z_2 z_6 z_7 + z_1 z_5 z_6 z_7 = 0 \\ z_1 z_2 z_3 z_4 z_5 z_6 + z_1 z_2 z_5 z_6 z_7 + z_1 z_4 z_5 z_6 z_7 = 0 \\ z_1 z_2 z_3 z_4 z_5 z_6 z_7 + z_1 z_2 z_4 z_5 z_6 z_7 = 0 \\ z_1 z_2 z_3 z_4 z_5 z_6 z_7 - 1 = 0 \end{cases} \quad (20)$$

By construction of  $M$ , observe that all polynomials have the same power of  $z_0$ , so  $z_0$  can be factored out. Removing  $z_0$  from the initial form system, we find a solution

$$z_0 = t, \quad z_1 = -I, \quad z_2 = \frac{-1}{2} - \frac{I}{2}, \quad z_3 = -1, \quad z_4 = 1 + I, \quad z_5 = \frac{1}{2} + \frac{I}{2}, \quad z_6 = I, \quad z_7 = -1 - I \quad (21)$$

where  $I = \sqrt{-1}$ . This solution is a regular solution, i.e.: it has multiplicity one. We set  $z_0 = t$ , where  $t$  is the variable for the Puiseux series. In the computation of the second term, we assume the Puiseux series of the form

$$\begin{cases} z_0 = t \\ z_1 = -I + c_1 t \\ z_2 = \frac{-1}{2} - \frac{I}{2} + c_2 t \\ z_3 = -1 + c_3 t \\ z_4 = 1 + I + c_4 t \\ z_5 = \frac{1}{2} + \frac{I}{2} + c_5 t \\ z_6 = I + c_6 t \\ z_7 = (-1 - I) + c_7 t \end{cases} \quad (22)$$

We first transform the cyclic 8-roots system  $C_8(\mathbf{x}) = \mathbf{0}$  using the coordinate transformation given by (19) and then substitute the assumed series form into this new system. Since the next term in the series is of form  $c_i t^1$ , we collect all the coefficients of  $t^1$  and solve the linear system of equations.

The second term in the Puiseux series expansion for the cyclic 8-root system, in the particular direction  $\mathbf{v}$ , has the form

$$\begin{cases} z_0 = t \\ z_1 = -I + (-1 - I)t \\ z_2 = \frac{-1}{2} - \frac{I}{2} + \frac{1}{2}t \\ z_3 = -1 \\ z_4 = 1 + I - t \\ z_5 = \frac{1}{2} + \frac{I}{2} - \frac{1}{2}t \\ z_6 = I + (1 + I)t \\ z_7 = (-1 - I) + t \end{cases} \quad (23)$$

Because of the regularity of the solution of the initial form system and the second term of the Puiseux series, we have a symbolic-numeric representation of a quadratic solution curve.

If we place the same pretropism on another row in the unimodular matrix, then we can develop the same curve starting at a different coordinate plane. This move is useful if the solution curve would not be in general position with respect to the first coordinate plane.

For symmetric polynomial systems, we apply the permutations to the pretropism, the initial form systems, and its solutions to find Puiseux series for different solution curves, related to the generating pretropism by symmetry.

In certain instances, the first term in the Puiseux series satisfies the entire system. We illustrate this situation by taking the pretropisms  $\mathbf{v} = (1, -1, 1, -1, 1, -1, 1, -1)$  of the cyclic 8-roots system and compute its initial form:

$$\text{in}_{\mathbf{v}}(C_8)(\mathbf{x}) = \begin{cases} x_1 + x_3 + x_5 + x_7 = 0 \\ x_0x_1 + x_0x_7 + x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_6 + x_6x_7 = 0 \\ x_0x_1x_7 + x_1x_2x_3 + x_3x_4x_5 + x_5x_6x_7 = 0 \\ x_0x_1x_2x_3 + x_0x_1x_2x_7 + x_0x_1x_6x_7 + x_0x_5x_6x_7 \\ + x_1x_2x_3x_4 + x_2x_3x_4x_5 + x_3x_4x_5x_6 + x_4x_5x_6x_7 = 0 \\ x_0x_1x_2x_3x_7 + x_0x_1x_5x_6x_7 + x_1x_2x_3x_4x_5 + x_3x_4x_5x_6x_7 = 0 \\ x_0x_1x_2x_3x_4x_5 + x_0x_1x_2x_3x_4x_7 + x_0x_1x_2x_3x_6x_7 + x_0x_1x_2x_5x_6x_7 \\ + x_0x_1x_4x_5x_6x_7 + x_0x_3x_4x_5x_6x_7 + x_1x_2x_3x_4x_5x_6 + x_2x_3x_4x_5x_6x_7 = 0 \\ x_0x_1x_2x_3x_4x_5x_7 + x_0x_1x_2x_3x_5x_6x_7 + x_0x_1x_3x_4x_5x_6x_7 + x_1x_2x_3x_4x_5x_6x_7 = 0 \\ x_0x_1x_2x_3x_4x_5x_6x_7 - 1 = 0 \end{cases} \quad (24)$$

The coordinate transformation for the initial form system (24) is given by the unimodular matrix

$$M = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The coordinate transformation  $\mathbf{x} = \mathbf{z}^M$  yields  $x_0 = z_0$ ,  $x_1 = z_1/z_0$ ,  $x_2 = z_0z_2$ ,  $x_3 = z_3/z_0$ ,  $x_4 = z_0z_4$ ,  $x_5 = z_5/z_0$ ,  $x_6 = z_0z_6$ ,  $x_7 = z_7/z_0$ . Applying the coordinate transforma-

tion to the initial form system (24) gives

$$\text{in}_{\mathbf{v}}(C_8)(\mathbf{x} = \mathbf{z}^M) = \begin{cases} z_1 + z_3 + z_5 + z_7 = 0 \\ z_1 z_2 + z_2 z_3 + z_3 z_4 + z_4 z_5 + z_5 z_6 + z_6 z_7 + z_1 + z_7 = 0 \\ z_1 z_2 z_3 + z_3 z_4 z_5 + z_5 z_6 z_7 + z_1 z_7 = 0 \\ z_1 z_2 z_3 z_4 + z_2 z_3 z_4 z_5 + z_3 z_4 z_5 z_6 + z_4 z_5 z_6 z_7 + z_1 z_2 z_3 \\ + z_1 z_2 z_7 + z_1 z_6 z_7 + z_5 z_6 z_7 = 0 \\ z_1 z_2 z_3 z_4 z_5 + z_3 z_4 z_5 z_6 z_7 + z_1 z_2 z_3 z_7 + z_1 z_5 z_6 z_7 = 0 \\ z_1 z_2 z_3 z_4 z_5 z_6 + z_2 z_3 z_4 z_5 z_6 z_7 + z_1 z_2 z_3 z_4 z_5 + z_1 z_2 z_3 z_4 z_7 \\ + z_1 z_2 z_3 z_6 z_7 + z_1 z_2 z_5 z_6 z_7 + z_1 z_4 z_5 z_6 z_7 + z_3 z_4 z_5 z_6 z_7 = 0 \\ z_1 z_2 z_3 z_4 z_5 z_6 z_7 + z_1 z_2 z_3 z_4 z_5 z_7 + z_1 z_2 z_3 z_5 z_6 z_7 + z_1 z_3 z_4 z_5 z_6 z_7 = 0 \\ z_1 z_2 z_3 z_4 z_5 z_6 z_7 - 1 = 0 \end{cases} \quad (25)$$

The initial form system (25) has 72 solutions. Among the 72 solutions, a solution of the form

$$z_0 = t, \quad z_1 = -1, \quad z_2 = I, \quad z_3 = -I, \quad z_4 = -1, \quad z_5 = 1, \quad z_6 = -I, \quad z_7 = I, \quad (26)$$

here expressed in the original coordinates,

$$x_0 = t, \quad x_1 = -1/t, \quad x_2 = It, \quad x_3 = -I/t, \quad x_4 = -t, \quad x_5 = 1/t, \quad x_6 = -It, \quad x_7 = I/t \quad (27)$$

satisfies the cyclic 8-roots entirely. Applying the cyclic permutation of this solution set we can obtain the remaining 7 solution sets, which also satisfy the cyclic 8-roots system.

In [23], a formula for the degree of the curve was derived, based on the coordinates of the tropism and the number of initial roots for the same tropism, and is given in the definition below.

**Definition 6.1** (Branch Degree). Let  $\mathbf{v} = (v_1, v_2, \dots, v_n)$  be a tropism and let  $R$  be the set of initial roots of the initial form system  $\text{in}_{\mathbf{v}}(f)(\mathbf{z})$ . Then the degree of the branch is

$$\#R \times \left| \max_{i=1}^n v_i - \min_{i=1}^n v_i \right|. \quad (28)$$

We apply Definition 6.1 to the Puiseux series developments for the cyclic 8-roots problem and obtain 144 as the known degree of the space curve of the one dimensional solution set.

$(1, -1, 1, -1, 1, -1, 1, -1)$	$8 \times 2 = 16$
$(1, -1, 0, 1, 0, 0, -1, 0) \rightarrow (1, 0, 0, -1, 0, 1, -1, 0)$	$8 \times 2 + 8 \times 2 = 32$
$(1, 0, -1, 0, 0, 1, 0, -1) \rightarrow (1, 0, -1, 1, 0, -1, 0, 0)$	$8 \times 2 + 8 \times 2 = 32$
$(1, 0, -1, 1, 0, -1, 0, 0) \rightarrow (1, 0, -1, 0, 0, 1, 0, -1)$	$8 \times 2 + 8 \times 2 = 32$
$(1, 0, 0, -1, 0, 1, -1, 0) \rightarrow (1, -1, 0, 1, 0, 0, -1, 0)$	$8 \times 2 + 8 \times 2 = 32$
	<b>TOTAL = 144</b>

Table 2: Tropisms, cyclic permutations, and degrees for the cyclic 8 solution curve.

Using the same polyhedral method we can find all the isolated solutions of the cyclic 8-roots system.

In the next few paragraphs, we address, informally, some of the issues related to the time complexity of the procedure for computation of the second term.

In the direction of tropism  $(1, -1, 0, 1, 0, 0, -1, 0)$ , there exists a second term in the Puiseux series. The procedure solves the initial form system, which yields 40 solutions, and checks whether the first term satisfies the cyclic 8-roots system. Then it proceeds to construct and compute the second term in the Puiseux series. The total time required is 35.5 seconds, which includes 28 milliseconds that PHCpack needed to solve the initial form system.

For the tropism  $(1, -1, 1, -1, 1, -1, 1, -1)$  there is no second term in the Puiseux series. In this direction, the first term solves the entire cyclic 8-roots system. Hence, the procedure for construction and computation of the second term does not run. It takes PHCpack 12 seconds to solve the initial form system, whose solution set consists of 509 solutions. Determining that there is no second term for the 509 solutions, takes 199 seconds.

Given their numbers of solutions, the ratio for time comparison is given by  $\frac{509}{40} = 12.725$ . However, given that for tropisms  $(1, -1, 0, 1, 0, 0, -1, 0)$  the procedure for construction and computation of the second term does run, unlike for tropism  $(1, -1, 1, -1, 1, -1, 1, -1)$ , the ratio for time comparison is not precise enough. A more accurate ratio for comparison is  $\frac{199}{35} \approx 5.686$ .

## 6.2 cyclic 12-roots

The generating solutions to the quadratic space curve solutions of the cyclic 12-roots problem are in Table 3. As the result in the Table 3 is given in the transformed coordinates, we return the solutions to the original coordinates. For any solution generator  $(r_1, r_2, \dots, r_{11})$  in Table 3:

$$\begin{aligned} z_0 = t, \quad z_1 = r_1, \quad z_2 = r_2, \quad z_3 = r_3, \quad z_4 = r_4, \quad z_5 = r_5, \\ z_6 = r_6, \quad z_7 = r_7, \quad z_8 = r_8, \quad z_9 = r_9, \quad z_{10} = r_{10}, \quad z_{11} = r_{11} \end{aligned} \tag{29}$$

and turning to the original coordinates we obtain

$$\begin{aligned} x_0 = t, \quad x_1 = r_1/t, \quad x_2 = r_2t, \quad x_3 = r_3/t, \quad x_4 = r_4t, \quad x_5 = r_5/t \\ x_6 = r_6t, \quad x_7 = r_7/t, \quad x_8 = r_8t, \quad x_9 = r_9/t, \quad x_{10} = r_{10}t, \quad x_{11} = r_{11}/t \end{aligned} \tag{30}$$

Applying Definition 6.1, we see that all space curves are quadrics. Compared to [16], we arrive at this result without the application of any factorization methods.

## 7 Concluding Remarks

Inspired by an effective proof of the fundamental theorem of tropical algebraic geometry, we outlined in this paper a new polynomial method to compute Puiseux series expansions for solution curves of polynomial systems. The main advantage of the new approach is the capability to exploit permutation symmetry.

For our current preliminary implementation we relied on `cddlib` for the pretropisms, the blackbox solver of PHCpack for solving the initial form systems, and Sage for the manipulations of the Puiseux series. Because of the many spurious normals caused by the Cayley embedding, the pretropism calculation can still be improved significantly. Often it may happen that the

initial form systems are sparser than the original system and are thus easier to solve. Another improvement to the new approach would be to replace a blackbox solver by a recursive application of the new approach to the initial form systems.

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$z_1$	$z_2$	$z_3$	$z_4$	$z_5$	$z_6$	$z_7$	$z_8$	$z_9$	$z_{10}$	$z_{11}$
$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	1	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	-1	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	-1	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$
$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	-1	1	1	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	-1	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	1	-1	-1	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$
$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	1	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	-1	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	-1	$-\frac{1}{2}$
$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	-1	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	1	-1	-1	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	1	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	-1	1
$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	-1	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	-1	$\frac{1}{2}$	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	1	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$
1	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	-1	1	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	-1	-1	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	1	-1	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$
$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	-1	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$
$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	-1	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	1	1	-1	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	1	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	-1	-1
1	-1	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	-1	-1	1	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$
$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	-1	$\frac{1}{2}$	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$
$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	1	-1	$\frac{1}{2}$	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	-1
$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	1	1	-1	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	-1	-1
$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	-1	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$
$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	-1	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$
$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	-1	-1	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	1	-1	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	1	1	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	-1
$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	-1	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	-1	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	1	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$
1	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	-1	-1	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$
-1	-1	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	-1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$
-1	$\frac{1}{2}$	1	1	$\frac{1}{2}$	-1	1	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	-1	-1	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$
$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$	1	$-\frac{1}{2}$	-1	$\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} + \frac{\sqrt{3}}{2}I$	$-\frac{1}{2} - \frac{\sqrt{3}}{2}I$	-1	$\frac{1}{2} - \frac{\sqrt{3}}{2}I$

Table 3: Generators of the roots of the initial form system  $\text{in}_{\mathbf{v}}(C_{12})(\mathbf{x}) = \mathbf{0}$  with the tropism  $\mathbf{v} = (+1, -1, +1, -1, +1, -1, +1, -1, +1, -1, +1, -1)$  in the transformed  $\mathbf{z}$  coordinates. Every solution defines a solution curve of the cyclic 12-roots system.