Abstract

We present a polyhedral algorithm to manipulate positive dimensional solution sets. Using facet normals to Newton polytopes as pretropisms, we focus on the first two terms of a Puiseux series expansion. The leading powers of the series are computed via the tropical prevariety. This polyhedral algorithm is well suited for exploitation of symmetry, when it arises in systems of polynomials. Initial form systems with pretropisms in the same group orbit are solved only once, allowing for a systematic filtration of redundant data. Computations with cddlib, Gfan, PHCpack, and Sage are illustrated on cyclic $n$-roots polynomial systems.

Keywords. Algebraic set, Backelin’s Lemma, cyclic $n$-roots, initial form, Newton polytope, polyhedral method, polynomial system, Puiseux series, symmetry, tropism, tropical prevariety.

1 Introduction

We consider a polynomial system $f(x) = 0$, $x = (x_0, x_1, \ldots, x_{n-1})$, $f = (f_1, f_2, \ldots, f_N)$, $f_i \in \mathbb{C}[x]$, $i = 1, 2, \ldots, N$. Although in many applications the coefficients of the polynomials are rational numbers, we allow the input system to have approximate complex numbers as coefficients. For $N = n$ (as many equations as unknowns), we expect in general to find only isolated solutions. In this paper we focus on cases $N \geq n$ where the coefficients are so special that $f(x) = 0$ has an algebraic set as solution.

Our approach is based on the following observation: If the solution set of $f(x) = 0$ has a space curve, then this space curve extends from $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ to infinity. In particular, the space curve intersects hyperplanes at infinity at isolated points. We start our series development of the space curve at these isolated points. Computing series developments for solutions of polynomial systems is a hybrid symbolic-numeric method, appropriate for inputs which consist
of approximate numbers (the coefficients) and exact data (the exponents). Formally we denote a polynomial \( f \in \mathbb{C}[x] \) as
\[
f(x) = \sum_{a \in A} c_a x^a, \quad c_a \in \mathbb{C}, \quad x^a = x_0^{a_0} x_1^{a_1} \cdots x_{n-1}^{a_{n-1}},
\]
and we call the set \( A \) of exponents the support of \( f \). The convex hull of \( A \) is the Newton polytope of \( f \). To each facet of the Newton polytope we may associate a hyperplane at infinity.

In this paper we will make various significant assumptions. First we assume that the algebraic sets we consider are reduced, that is: free of multiplicities. Moreover, an algebraic set of dimension \( d \) is in general position with respect to the first \( d \) coordinate planes. For example, we assume that a space curve is not contained in a plane perpendicular to the first coordinate axis. Thirdly, we assume the algebraic set of dimension \( d \) to intersect the first \( d \) coordinate planes at regular solutions. For space curves, the special role of \( x_0 \) is reflected in the normal form of the Puiseux series:
\[
\begin{align*}
x_0 &= t^{v_0} \\
x_i &= t^{v_i}(y_t + z_t^{w_i}), & i = 1, 2, \ldots, n - 1,
\end{align*}
\]
where the leading powers \( v = (v_0, v_1, \ldots, v_{n-1}) \) equal to what is called a tropism. For algebraic sets of dimension \( d \), there is a \( d \)-dimensional polyhedral cone spanned by tropisms. The coordinates of the tropisms define the exponents of the leading terms in the \( d \) parameters of the series.

Our approach consists of two stages. The computation of the pretropisms, which are candidates for the leading powers of the Puiseux series, is followed by the computation of the leading coefficients and the second term of the Puiseux series, if the leading term of the series does not already entirely satisfy the system. The second term of the Puiseux series indicates the existence of a space curve. If the system is invariant to permutation of the variables, then it suffices to compute only the generators of the solution orbits. We then develop the Puiseux series only at the generators. Although our approach is directed at general algebraic sets, our approach of exploiting symmetry applies also to the computation of all isolated solutions. Our main example is one family of polynomial systems, the cyclic \( n \)-roots system.

Related Work. Our approach is inspired by the constructive proof of the fundamental theorem of tropical algebraic geometry in [31] (an alternative proof is in [37]) and related to finiteness proofs in celestial mechanics [26], [29]. The initial form systems allow the elimination of variables with the application of coordinate transformations, an approach presented in [28] and related to the application of the Smith normal form in [24]. The complexity of polyhedral homotopies is studied in [32] and generalized to affine solutions in [27]. Generalizations of the Newton-Puiseux theorem [41], [55], can be found in [5], [7], [35], [36], [42], and [44]. A symbolic-numeric computation of Puiseux series is described in [38], [39], and [40]. Algebraic approaches to exploit symmetry are [13], [20], [22], and [47]. The cyclic \( n \)-roots problem is a benchmark for polynomial system solvers, see e.g: [9], [13], [14], [16], [17], [18], [20], [34], [47], and relevant to operator algebras [10], [25], [51]. Our results on cyclic 12-roots correspond to [43].

Our Contributions. This paper is a thorough revision of the unpublished preprint [2], originating in the dissertation of the first author [1], which extended [3] from the plane to space curves. In [4] we gave a tropical version of Backelin’s lemma in case \( n = m^2 \), in this paper we generalize to the case \( n = \ell m^2 \). Our approach improves homotopies to find all isolated solutions. Exploiting
symmetry we compute only the generating cyclic $n$-roots, more efficiently than the symmetric polyhedral homotopies of [54].

2 Initial Forms, Cyclic $n$-roots, and Backelin’s Lemma

In this section we introduce our approach on the cyclic 4-roots problem. For this problem we can compute an explicit representation for the solution curves. This explicit representation as monomials in the independent parameters for positive dimensional solution sets generalizes into the tropical version of Backelin’s lemma.

2.1 The Cyclic $n$-roots Problem

For $n = 3$, the cyclic $n$-roots system originates naturally from the elementary symmetric functions in the roots of a cubic polynomial. For $n = 4$, the system is

$$f(x) = \begin{cases} 
  x_0 + x_1 + x_2 + x_3 = 0 \\
  x_0x_1 + x_1x_2 + x_2x_3 + x_3x_0 = 0 \\
  x_0x_1x_2 + x_1x_2x_3 + x_2x_3x_0 + x_3x_0x_1 = 0 \\
  x_0x_1x_2x_3 - 1 = 0.
\end{cases} \tag{3}$$

The permutation group which leaves the equations invariant is generated by $(x_0, x_1, x_2, x_3) \rightarrow (x_1, x_2, x_3, x_0)$ and $(x_0, x_1, x_2, x_3) \rightarrow (x_3, x_2, x_1, x_0)$. In addition, the system is equi-invariant with respect to the action $(x_0, x_1, x_2, x_3) \rightarrow (x_0^1, x_1^1, x_2^1, x_3^1)$.

**Definition 2.1.** Let $\mathbf{v} \neq 0$, denote $\langle \mathbf{a}, \mathbf{v} \rangle = a_0v_0 + a_1v_1 + \cdots + a_nv_n$, and let $f$ be a polynomial supported on $A$. Then, the initial form of $f$ in the direction of $\mathbf{v}$ is

$$\text{in}_\mathbf{v}(f) = \sum_{\mathbf{a} \in A} c_\mathbf{a}x^\mathbf{a}, \quad \text{for } f = \sum_{\mathbf{a} \in A} c_\mathbf{a}x^\mathbf{a} \quad \text{where } m = \min\{ \langle \mathbf{a}, \mathbf{v} \rangle | \mathbf{a} \in A \}. \tag{4}$$

The initial form of a system with polynomials in $f = (f_1, f_2, \ldots, f_N)$ in the direction of $\mathbf{v}$ is denoted by $\text{in}_\mathbf{v}(f) = (\text{in}_\mathbf{v}(f_1), \text{in}_\mathbf{v}(f_2), \ldots, \text{in}_\mathbf{v}(f_N))$.

The notation $\text{in}_\mathbf{v}(f)$ follows [50], where $\mathbf{v}$ represents a weight vector to order monomials. In [12] and [33], initial form systems are called truncated systems.

With $\mathbf{v} = (+1, -1, +1, -1)$, there is a unimodular coordinate transformation $M$, denoted by $x = y^M$:

$$\text{in}_\mathbf{v}(f)(x) = \begin{cases} 
  x_1 + x_3 = 0 \\
  x_0x_1 + x_1x_2 + x_2x_3 + x_3x_0 = 0 \\
  x_1x_2x_3 + x_3x_0x_1 = 0 \\
  x_0x_1x_2x_3 - 1 = 0
\end{cases} \quad M = \begin{bmatrix} +1 & -1 & +1 & -1 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix} \quad x = y^M : \begin{cases} 
  x_0 = y_0^{+1} \\
  x_1 = y_0^{-1}y_1 \\
  x_2 = y_0^{-1}y_2 \\
  x_2 = y_0^{-1}y_3
\end{cases} \tag{5}$$
The system \( \text{inv}(f)(y) = 0 \) has two solutions. These two solutions are the leading coefficients in the Puiseux series. In this case, the leading term of the series vanishes entirely at the system so we write two solution curves as \((t, -t^{-1}, -t, -t^{-1})\) and \((t, t^{-1}, -t, -t^{-1})\). To compute the degree of the two solution curves, we take a random hyperplane in \( \mathbb{C}^4 \): \( c_0 x_0 + c_1 x_1 + c_2 x_2 + c_3 x_3 + c_5 = 0 \), \( c_i \in \mathbb{C}^* \). Then the number of points on the curve and on the random hyperplane equals the degree of the curve. Substituting the representations we obtained for the curves into the random hyperplanes gives a quadratic polynomial in \( t \) (after clearing the denominator \( t^{-1} \)), so there are two quadric curves of cyclic 4-roots.

### 2.2 A Tropical Version of Backelin’s Lemma

In [4], we gave an explicit representation for the solution sets of cyclic \( n \)-roots, in case \( n = m^2 \), for any natural number \( m \geq 2 \). Below we state Backelin’s Lemma [6], in its tropical form.

**Lemma 2.2** (Tropical Version of Backelin’s Lemma). For \( n = m^2 \ell \), where \( \ell \in \mathbb{N} \setminus \{0\} \) and \( \ell \) is no multiple of \( k^2 \), for \( k \geq 2 \), there is an \((m - 1)\)-dimensional set of cyclic \( n \)-roots, represented exactly as

\[
\begin{align*}
x_{km+0} & = u^k t_0 \\
x_{km+1} & = u^k t_0 t_1 \\
x_{km+2} & = u^k t_0 t_1 t_2 \\
& \quad \vdots \\
x_{km+m-2} & = u^k t_0 t_1 t_2 \cdots t_{m-2} \\
x_{km+m-1} & = \gamma u^k t_0^{m+1} t_1^{m+2} \cdots t_{m-3}^{m-2} t_{m-2}
\end{align*}
\]

for \( k = 0, 1, 2, \ldots, m - 1 \), free parameters \( t_0, t_1, \ldots, t_{m-2} \), constants \( u = e^{i\pi \alpha} \), \( \gamma = e^{i\pi \beta} \), with \( \beta = (\alpha \mod 2) \), and \( \alpha = m(\ell - 1) \).

**Proof.** By performing the change of variables \( y_0 = t_0 \), \( y_1 = t_0 t_1 \), \( y_2 = t_0 t_1 t_2 \), \ldots, \( y_{m-2} = t_0 t_1 t_2 \cdots t_{m-2} \), \( y_{m-1} = \gamma t_0^{m+1} t_1^{m+2} \cdots t_{m-3}^{m-2} t_{m-2} \), the solution (6) can be rewritten as

\[
x_{km+j} = u^k y_j, \quad j = 0, 1, \ldots, m - 1.
\]

The solution (7) satisfies the cyclic \( n \)-roots system by plain substitution as in the proof of [19, Lemma 1.1], whenever the last equation \( x_0 x_1 x_2 \cdots x_{n-1} - 1 = 0 \) of the cyclic \( n \)-roots problem can also be satisfied.

We next show that we can always satisfy the equation \( x_0 x_1 x_2 \cdots x_{n-1} - 1 = 0 \) with our solution. First, we perform an additional change of coordinates to separate the \( \gamma \) coefficient. We let \( y_0 = Y_0 \), \( y_1 = Y_1 \), \ldots, \( y_{m-2} = Y_{m-2} \), \( y_{m-1} = \gamma Y_{m-1} \). Then on substitution of (7) into \( x_0 x_1 x_2 \cdots x_{n-1} - 1 = 0 \), we get
\begin{align*}
\left(\gamma_{m\ell} u^0 u^0 \cdots u^0 u^1 \cdots u^1 u^{m-1} u^{m-1} \cdots u^{m-1}\right) Y_0^{m\ell} Y_1^{m\ell} Y_2^{m\ell} \cdots Y_{m-2}^{m\ell} Y_{m-1}^{m\ell} - 1 &= 0 \\
\left(\gamma_{m\ell} u^{0m} u^1 m u^2 m \cdots u^{(m-1)m} m Y_0^{m\ell} Y_1^{m\ell} Y_2^{m\ell} \cdots Y_{m-2}^{m\ell} Y_{m-1}^{m\ell} - 1 &= 0 \\
\left(\gamma_{m\ell} u^{m(0+1+2+\cdots+(m-1))} m Y_0^{m\ell} Y_1^{m\ell} Y_2^{m\ell} \cdots Y_{m-2}^{m\ell} Y_{m-1}^{m\ell} - 1 &= 0 \\
\left(\gamma_{m\ell} u^{m^2 (m-1)} m Y_0^{m\ell} Y_1^{m\ell} Y_2^{m\ell} \cdots Y_{m-2}^{m\ell} Y_{m-1}^{m\ell} - 1 &= 0 \\
\gamma_{m\ell} \left(\frac{u^{m(m-1)}}{2}\right)^{m\ell} (Y_0 Y_1 Y_2 \cdots Y_{m-2} Y_{m-1})^{m\ell} - 1 &= 0. \\
\left(\gamma u^{\frac{m(m-1)}{2}} \right)^{m\ell} - 1 &= 0. \\
\end{align*}

The last equation in (8) has now the same form as in [19, Lemma 1.1]. We are done if we can satisfy it. We next show that it can always be satisfied with our solution.

Since all the tropisms in the cone add up to zero, the product \((Y_0 Y_1 Y_2 \cdots Y_{m-2} Y_{m-1})\), which consists of free parameter combinations, equals to 1. Since \((Y_0 Y_1 Y_2 \cdots Y_{m-2} Y_{m-1}) = 1\), we are left with

\begin{equation}
\left(\gamma u^{\frac{m(m-1)}{2}} \right)^{m\ell} - 1 = 0. \\
\end{equation}

We distinguish two cases:

1. \(\gamma = 1\), implied by \(m\) is even, \(\ell\) is odd) or \((m\) is odd, \(\ell\) is odd) or \((m\) is even, \(\ell\) is even).

To show that (9) is satisfied, we rewrite (9):

\begin{align*}
\left(u^{\frac{m(m-1)}{2}}\right)^{m\ell} - 1 &= 0 \\
\left(u^{\frac{m^2 (m-1)}{2}}\right) - 1 &= 0 \\
\left(\frac{u^{m\ell}}{2}\right)^{m(m-1)} - 1 &= 0,
\end{align*}

which is satisfied by \(u = e^{i\frac{2\pi}{m\ell}}\) and \(m(m\ell - 1)\) being even.

2. \(\gamma \neq 1\), implied by \((m\) is odd, \(\ell\) is even).

To show that our solution satisfies (9), we rewrite (9):

\begin{align*}
\left(\gamma u^{\frac{m(m-1)}{2}} \right)^{m\ell} - 1 &= 0 \\
\left(\gamma u^{\frac{m^2 \ell}{2}} u^{-m} \right)^{m\ell} - 1 &= 0 \\
\left(\gamma u^{m\ell} \right)^{m} u^{-m} - 1 &= 0.
\end{align*}

Since \(u = e^{i\frac{2\pi}{m\ell}}, u^{m\ell} = 1\), we can simplify (11) further.
\[(\gamma \frac{u - m}{m\ell})^{1} - 1 = 0\]
\[(\frac{e^{i\pi}}{m\ell} (\frac{u}{m\ell})^{1/2})^{m\ell} - 1 = 0\]
\[(\frac{e^{i\pi}}{m\ell} (\frac{u}{m\ell})^{-1/2} m\ell - 1 = 0\]
\[(\frac{e^{i\pi}}{m\ell} (\frac{e^{-i\pi}}{m\ell})^{m\ell} - 1 = 0\]
\[(e^{i\pi} e^{-i\pi m}) - 1 = 0\]
\[(e^{i\pi} (1 - m)\pi) - 1 = 0\].

Since \(m\) is odd, we can write \(m = 2j + 1\), for some \(j\). The last equation of (12) has the form
\[(e^{i\pi (1 - m)\pi}) - 1 = 0\]
\[(e^{i\pi (1 - (2j + 1))\pi}) - 1 = 0\]
\[(e^{i\pi (-2j)\pi}) - 1 = 0\].

Since \(e^{i\pi (-2j)\pi} = 1\), for any \(j\), the equation \(e^{i\pi (-2j)\pi} - 1 = 0\) is satisfied, implying (9).

\[\square\]

3 Exploiting Symmetry

We illustrate the exploitation of permutation symmetry on the cyclic 5-roots system. Adjusting polyhedral homotopies to exploit the permutation symmetry for this system was presented in [54].

3.1 The Cyclic 5-roots Problem

The mixed volume for the cyclic 5-roots system is 70, which equals the exact number of roots. If we consider the first four equations of the cyclic 5-roots system \(C_5(x) = 0\), then we have solution curves. Consider the first four equations of \(C_5(x) = 0\):

\[
\begin{align*}
  x_0 + x_1 + x_2 + x_3 + x_4 &= 0 \\
  x_0x_1 + x_0x_4 + x_1x_2 + x_2x_3 + x_3x_4 &= 0 \\
  x_0x_1x_2 + x_0x_1x_4 + x_0x_3x_4 + x_1x_2x_3 + x_2x_3x_4 &= 0 \\
  x_0x_1x_2x_3 + x_0x_1x_2x_4 + x_0x_1x_3x_4 + x_0x_2x_3x_4 + x_1x_2x_3x_4 &= 0
\end{align*}
\]

where \(v = (1, 1, 1, 1, 1)\). As the first four equations of \(C_5\) are homogeneous, the first four equations of \(C_5\) coincide with the first four equations of \(\text{im}_v(C_5)(x) = 0\). Because these four equations are homogeneous, we have lines of solutions. After computing representations for the solution lines, we find the solutions to the original cyclic 5-roots problem intersecting the solution lines with the hypersurface defined by the last equation. In this intersection, the exploitation of the symmetry is straightforward.

The unimodular matrix with \(v = (1, 1, 1, 1, 1)\) and its corresponding coordinate transformation
The total number of solutions is 70, as indicated by the mixed volume computation. Existence of a positive dimensional solution set either. We look for $v$ so that $\text{in}_v(f)(x) = 0$ has solutions in $(\mathbb{C}^*)^n$. A nonzero vector $v$ is a pretropism for the system $f(x) = 0$ if $\# \text{in}_v(f_k) \geq 2$ for all $k$ ranging from 1 to $N$. Every tropism is a pretropism, but not every pretropism is a tropism, as pretropisms depend only on supports $A = (A_1, A_2, \ldots, A_N)$ of $f$.

### 3.2 A General Approach

That the first $n - 1$ equations of cyclic $n$-roots system give explicit solution lines is exceptional. In general, we can use the leading term of the Puiseux series to compute witness sets \[46\] for the space curves defined by the first $n - 1$ equations. Then via the diagonal homotopy \[45\] we can intersect the space curves with the rest of the system. While the direct exploitation of symmetry with witness sets is not possible, with the Puiseux series we can pick out the generating space curves.

### 4 Computing Pretropisms

Following from the second theorem of Bernshtein \[8\], the Newton polytopes may be in general position and no normals to at least one edge of every Newton polytope exists. In that case, there does not exist a positive dimensional solution set either. We look for $v$ so that $\text{in}_v(f)(x) = 0$ has solutions in $(\mathbb{C}^*)^n$. A nonzero vector $v$ is a pretropism for the system $f(x) = 0$ if $\# \text{in}_v(f_k) \geq 2$ for all $k$ ranging from 1 to $N$. Every tropism is a pretropism, but not every pretropism is a tropism, as pretropisms depend only on supports $A = (A_1, A_2, \ldots, A_N)$ of $f$. 

\[
\begin{align*}
M &= \begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
\end{bmatrix} \\
x &= z^M : \begin{cases}
x_0 = z_0 \\
x_1 = z_0 z_1 \\
x_2 = z_0 z_2 \\
x_3 = z_0 z_3 \\
x_4 = z_0 z_4 \\
\end{cases}
\end{align*}
\]

Applying $x = z^M$ to the initial form system (14) gives

\[
\text{in}_v(C_8)(x = z^M) = \begin{cases}
z_1 + z_2 + z_3 + z_4 + 1 = 0 \\
z_1 z_2 + z_2 z_3 + z_3 z_4 + z_1 + z_4 = 0 \\
z_1 z_2 z_3 + z_2 z_3 z_4 + z_1 z_2 + z_1 z_4 + z_3 z_4 = 0 \\
z_1 z_2 z_3 z_4 + z_1 z_2 z_3 + z_1 z_2 z_4 + z_1 z_3 z_4 + z_2 z_3 z_4 = 0
\end{cases}
\]

The system (16) has 14 isolated solutions of the form $z_1 = c_1$, $z_2 = c_2$, $z_3 = c_3$, $z_4 = c_4$. If we let $z_0 = t$, in the original coordinates we have

\[
\begin{align*}
x_0 &= t, \quad x_1 = t c_1, \quad x_2 = t c_2, \quad x_3 = t c_3, \quad x_4 = t c_4
\end{align*}
\]

as representations for the 14 solution lines.

Substituting (17) into the omitted equation $x_0 x_1 x_2 x_3 x_4 - 1 = 0$, yields a univariate polynomial in $t$ of the form $k t^5 - 1 = 0$, where $k$ is a constant. Among the 14 solutions, 10 are of the form $t^5 - 1$. They account for 2 $\times$ 5 = 10 solutions. There are two solutions of the form $(-122.99186938124345) t^5 - 1$, accounting for 2 $\times$ 5 = 10 solutions and an additional two solutions are of the form $(-0.0081306187557833118) t^5 - 1$ accounting for 2 $\times$ 5 = 10 remaining solutions. The total number of solutions is 70, as indicated by the mixed volume computation. Existence of additional symmetry, which can be exploited, can be seen in the relationship between the coefficients of the quintic polynomial, i.e. $\frac{1}{(-122.99186938124345)} \approx -0.0081306187557833118$. 

\[
\text{in}_v(C_8)(x = z^M) = \begin{cases}
z_1 + z_2 + z_3 + z_4 + 1 = 0 \\
z_1 z_2 + z_2 z_3 + z_3 z_4 + z_1 + z_4 = 0 \\
z_1 z_2 z_3 + z_2 z_3 z_4 + z_1 z_2 + z_1 z_4 + z_3 z_4 = 0 \\
z_1 z_2 z_3 z_4 + z_1 z_2 z_3 + z_1 z_2 z_4 + z_1 z_3 z_4 + z_2 z_3 z_4 = 0
\end{cases}
\]
In this section we describe two approaches to compute pretropisms. The first approach applies cddlib [21] on the Cayley embedding. Algorithms to compute tropical varieties are described in [11] and implemented in Gfan [30]. The second approach is the application of tropical_intersection of Gfan.

4.1 Using the Cayley Embedding

The Cayley trick formulates a resultant as a discriminant as in [23, Proposition 1.7, page 274]. We follow the geometric description of [49], see also [15, §9.2]. The Cayley embedding $E_A$ of $A$ is

$$E_A = (A_1 \times \{0\}) \cup (A_2 \times \{e_1\}) \cup \cdots \cup (A_N \times \{e_{N-1}\})$$

(18)

where $e_k$ is the $k$th $(N - 1)$-dimensional unit vector. We call the convex hull of the Cayley embedding the Cayley polytope $\text{conv}(E_A)$ Enumerating all facet normals to $\text{conv}(E_A)$ yields all tropisms. Furthermore, tropisms for curves are normals to facets spanned by at least two points of each support. In case of surfaces of solutions, we will find cones spanned by tropisms for curves.

Running cddlib [21] to compute the H-representation of the Cayley polytope of the cyclic 8-roots problem yields 94 pretropisms. With symmetry we have 11 generators, displayed in Table 1.

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<td>39 \</td>
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<td>\</td>
<td>\</td>
<td>\</td>
</tr>
<tr>
<td>10. $(-1, 0, 1, 0, -1, 1, -1, 1)$ \</td>
<td>23 \</td>
<td>\</td>
<td>\</td>
<td>\</td>
<td>\</td>
</tr>
<tr>
<td>11. $(-1, 1, -1, 1, -1, 1, -1, 1)$ \</td>
<td>509 \</td>
<td>\</td>
<td>\</td>
<td>\</td>
<td>\</td>
</tr>
</tbody>
</table>

Table 1: Eleven pretropism generators of the cyclic 8-root problem, the number of solutions of the corresponding initial form systems, and the multidimensional cones they generate.

For the cyclic 9-roots problem, the computation of the facets of the Cayley polytope yield 276 pretropisms, with 17 generators: $(-2, 1, 1, -2, 1, 1, -2, 1, 1), (-1, -1, 2, -1, -1, 2, 1, -1, 2)$, $(-1, 0, 0, 0, 0, 1, -1, 1, 0), (-1, 0, 0, 0, 0, 1, 0, -1, 1), (-1, 0, 0, 0, 1, -1, 0, 1, 0), (-1, 0, 0, 1, -1, 1, 0, 0), (-1, 0, 0, 1, 0, -1, 0, 1), (-1, 0, 0, 0, 0, 1, 0, -1, 1), (-1, 0, 0, 0, 1, 0, -1, 1, 0), (-1, 0, 0, 0, 1, 0, -1, 1, 1), (-1, 0, 0, 0, 0, 0, -1, 0, 1), (-1, 0, 0, 0, 0, 0, 0, -1, 1), (-1, 0, 0, 0, 0, 0, 0, -1, 2). To get the structure of the two
dimensional cones, a second run of the Cayley embedding is needed on the smaller initial form systems defined by the pretropisms.

The computations for \( n = 8 \) and \( n = 9 \) finished in less than a second on one core of a 3.07Ghz Linux computer with 4Gb RAM. For the cyclic 12-roots problem, \texttt{cddlib} needed about a week to compute the 907,923 facets normals of the Cayley polytope. Although effective, the Cayley embedding becomes too inefficient for larger problems.

4.2 Using \texttt{tropical_intersection} of \texttt{Gfan}

The solution set of the cyclic 8 roots polynomial system consists of space curves. Therefore, all tropisms cones were generated by a single tropism. The computation of the tropical \textit{prevariety}, however, did not lead only to single pretropisms but also to cones of pretropisms. The cyclic 8-roots cones of pretropisms and their dimension are listed in Table 1. Since the one dimensional rays of pretropisms yielded initial form systems with isolated solutions and since all higher dimensional cones are spanned by those one dimensional rays, we can conclude that there are no higher dimensional algebraic sets, as any two dimensional surface degenerates to a curve if we consider only one tropism.

For the computation of the tropical prevariety, the Sage 5.7/Gfan function \texttt{tropical_intersection()} ran (with default settings without exploitation of symmetry) on an AMD Phenom II X4 820 processor with 6 GB of RAM, running GNU/Linux, see Table 2. As the dimension \( n \) increases so does the running time, but the relative cost factors are bounded by \( n \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>\text{seconds}</th>
<th>\text{hms format}</th>
<th>\text{factor}</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>16.37</td>
<td>16 s</td>
<td>1.0</td>
</tr>
<tr>
<td>9</td>
<td>79.36</td>
<td>1 m 19 s</td>
<td>4.8</td>
</tr>
<tr>
<td>10</td>
<td>503.53</td>
<td>8 m 23 s</td>
<td>6.3</td>
</tr>
<tr>
<td>11</td>
<td>3898.49</td>
<td>1 h 4 m 58 s</td>
<td>7.7</td>
</tr>
<tr>
<td>12</td>
<td>37490.93</td>
<td>10 h 24 m 50 s</td>
<td>9.6</td>
</tr>
</tbody>
</table>

Table 2: Time to compute the tropical prevarieties for cyclic \( n \)-roots with Sage 5.7/Gfan and the relative cost factors: for \( n = 12 \), it takes 9.6 times longer than for \( n = 11 \).

5 The Second Term of a Puiseux Series

In exceptional cases like the cyclic 4-roots problem where the first term of the series gives an exact solution or when we encounter solution lines like with the first four equations of cyclic 5-roots, we do not have to look for a second term of a series. In general, a pretropism \( v \) becomes a tropism if there is a Puiseux series with leading powers equal to \( v \). The leading coefficients of the series is a solution in \( \mathbb{C}^* \) of the initial form system \( \text{in}_v(f)(x) = 0 \). We solve the initial form systems with \texttt{PHCpack} [52]. For the computations of the series we use Sage [48].
5.1 Computing the Second Term

In our approach, the calculation of the second term in the Puiseux series is critical to decide whether a solution of an initial form system corresponds to an isolated solution at infinity of the original system, or whether it constitutes the beginning of a space curve. For sparse systems, we may not assume that the second term of the series is linear in \( t \). Trying consecutive powers of \( t \) will be wasteful for high degree second terms of particular systems. In this section we explain our algorithm to compute the second term in the Puiseux series.

A unimodular coordinate transformation \( \mathbf{x} = \mathbf{z}^M \) with \( M \) having as first row the vector \( \mathbf{v} \) turns the initial form system \( \text{in}_v(f)(\mathbf{x}) = \mathbf{0} \) into \( \text{in}_{e_1}(f)(\mathbf{z}) = \mathbf{0} \) where \( e_1 = (1, 0, \ldots, 0) \) equals the first standard basis vector. When \( \mathbf{v} \) has negative components, solutions of \( \text{in}_v(f)(\mathbf{x}) = \mathbf{0} \) that are at infinity (in the ordinary sense of having components equal to \( \infty \)) are turned into solutions in \( (\mathbb{C}^*)^n \) of \( \text{in}_{e_1}(f)(\mathbf{z}) = \mathbf{0} \).

**Proposition 5.1.** If the initial root does not satisfy the entire transformed polynomial system, then there must be at least one nonzero constant exponent \( a_i \) forming a monomial \( c_i t^{a_i} \).

**Proof.** Let \( \mathbf{z} = (z_0, z_1, \ldots, z_{n-1}) \) and \( \bar{\mathbf{z}} = (z_1, z_2, \ldots, z_{n-1}) \) denote variables after the unimodular transformation. Let \( (z_0 = t, z_1 = r_1, \ldots, z_{n-1} = r_{n-1}) \) be a regular solution at infinity and \( t \) the free variable.

The \( i \)th equation of the original system after the unimodular coordinate transformation has the form

\[
    f_i = z_0^{m_i}(P_i(\bar{\mathbf{z}}) + O(z_0)Q_i(\mathbf{z})) , \quad i = 1, 2, \ldots, N,
\]

where the polynomial \( P_i(\bar{\mathbf{z}}) \) consists of all monomials which form the initial form component of \( f_i \) and \( Q_i(\mathbf{z}) \) is a polynomial consisting of all remaining monomials of \( f_i \). After the coordinate transformation, we denote the series expansion as

\[
    \begin{cases} 
        z_0 = t \\
        z_i = r_i + k_i t^w, \quad i = 1, 2, \ldots, n - 1. 
    \end{cases}
\]

We first show that polynomial \( z_0^{m_i}P_i(\bar{\mathbf{z}}) \) cannot contain a monomial of the form \( c_i t^{a_i} \) on substitution of (20). The polynomial \( z_0^{m_i}P_i(\bar{\mathbf{z}}) \) is the initial form of \( f_i \), hence solution at infinity \( (z_0 = t, z_1 = r_1, z_2 = r_2, \ldots, z_{n-1} = r_{n-1}) \) satisfies \( z_0^{m_i}P_i(\bar{\mathbf{z}}) \) entirely. Substituting (20) into \( z_0^{m_i}P_i(\bar{\mathbf{z}}) \) eliminates all constants in \( t^{m_i}P_i(\bar{\mathbf{z}}) \). Hence, the polynomial \( P_i(t) = R_i(t^w) \) and, therefore, \( t^{m_i}P_i(t) = R_i(t^{w+m_i}) \).

We next show that polynomial \( Q_i(\mathbf{z}) \) contains a monomial \( c_i t^{a_i} \). The polynomial \( Q_i(\mathbf{z}) \) is rewritten:

\[
    z_0^{\alpha_i}Q_i(\bar{\mathbf{z}}) = z_0^{\alpha_i}T_{i0}(\bar{\mathbf{z}}) + z_0^{\alpha_i+1}T_{i1}(\bar{\mathbf{z}}) + \cdots.
\]

The polynomial \( Q_i(\mathbf{z}) = z_0^{\alpha_i}Q_i(\bar{\mathbf{z}}) \) consists of monomials which are not part of the initial form of \( f_i \). Hence, on substitution of solution at infinity (20), \( z_0^{\alpha_i}Q_i(\bar{\mathbf{z}}) = t^{a_i}Q_i(t) \) does not vanish entirely and it must contain monomials which are constants. Since \( Q_i(t) \) contains monomials which are constants, \( t^{a_i}Q_i(t) \) must contain a monomial of the form \( c_i t^{a_i} \).\[\square\]
If the initial root does not satisfy the original system, then we may have a second term in the Puiseux series. Assume the following general form of the series:

\[
\begin{align*}
    z_0 &= t \\
    z_i &= c_i^{(0)} + y_i t^w, \quad i = 1, 2, \ldots, n-1,
\end{align*}
\]  

(22)

where \(c_i^{(0)} \in \mathbb{C}^*\) are the coordinates of the initial root, \(y_i\) is the unknown coefficient of the second term \(t^w\), \(w > 0\). Note that only for some \(y_i\) nonzero values may exist. We are looking for the smallest \(w\) for which the linear system in the \(y_i\)'s admits a solution with at least one nonzero coordinate. Substituting (22) gives equations of the form

\[
\widetilde{c}_i^{(0)} a_i (1 + O(t)) + t^{w+b_i} \sum_{j=1}^{n} \gamma_{ij} y_j (1 + O(t)) = 0, \quad i = 1, 2, \ldots, n,
\]

(23)

for constant exponents \(a_i, b_i\) and constant coefficients \(\widetilde{c}_i^{(0)}\) and \(\gamma_{ij}\).

In the equations of (23) we truncate the \(O(t)\) terms and retain those equations with the smallest value of the exponents \(a_i\), because with the second term of the series solution we want to eliminate the lowest powers of \(t\) when we plug in the first two terms of the series in the system. This gives a condition on the value \(w\) of the unknown exponent of \(t\) in the second term. If there is no value for \(w\) so that we can match with \(w+b_i\) the minimal value of \(a_i\) for all equations where the same minimal value of \(a_i\) occurs, then there does not exist a second term and hence no space curve. Otherwise, with the matching value for \(w\) we obtain a linear system in the unknown \(y\) variables. If a solution to this linear system exists with at least one nonzero coordinate, then we have found a second term, otherwise, there is no space curve.

For an algebraic set of dimension \(d\), we have a polyhedral cone of \(d\) tropisms and we take any general vector \(v\) in this cone. Then we apply the method outlined above to compute the second term in the series in one parameter, in the direction of \(v\).

### 5.2 Series Developments for Cyclic 8-roots

We illustrate our approach on the cyclic 8-roots problem, denoted by \(C_8(x) = 0\) and take as pretropism \(v = (1, -1, 0, 1, 0, 0, -1, 0)\). The corresponding initial form system is

\[
\text{inv}_v(C_8)(x) = \left\{ \begin{array}{l}
    x_1 + x_6 = 0 \\
    x_1 x_2 + x_5 x_6 + x_6 x_7 = 0 \\
    x_4 x_5 x_6 + x_5 x_6 x_7 = 0 \\
    x_0 x_1 x_6 x_7 + x_4 x_5 x_6 x_7 = 0 \\
    x_0 x_1 x_2 x_6 x_7 + x_0 x_1 x_5 x_6 x_7 = 0 \\
    x_0 x_1 x_2 x_5 x_6 x_7 + x_0 x_1 x_4 x_5 x_6 x_7 + x_1 x_2 x_3 x_4 x_5 x_6 = 0 \\
    x_0 x_1 x_2 x_4 x_5 x_6 x_7 + x_1 x_2 x_3 x_4 x_5 x_6 x_7 = 0 \\
    x_0 x_1 x_2 x_3 x_4 x_5 x_6 x_7 - 1 = 0
\end{array} \right.
\]

(24)

A unimodular matrix \(M \in \mathbb{Z}^{n \times n}\) is an invertible integer matrix. We use unimodular matrices for coordinate transformations. Our unimodular matrix for this case equals the identity matrix, except for the first row, which is replaced by the tropism \((1, -1, 0, 1, 0, 0, -1, 0)\). Then the
corresponding unimodular coordinate transformation, denoted as \( x = z^M \), is defined as

\[
x_0 = z_0, \ x_1 = z_1/z_0, \ x_2 = z_2, \ x_3 = z_0z_3, \ x_4 = z_4, \ x_5 = z_5, \ x_6 = z_6/z_0, \ x_7 = z_7
\]  
(25)

Applying \( x = z^M \) to the initial form system (24) gives

\[
ineq(C_8)(x = z^M) = \begin{cases} 
z_1 + z_6 = 0 \\
z_1z_2 + z_5z_6 + z_6z_7 = 0 \\
z_1z_5z_6 + z_5z_6z_7 = 0 \\
z_1z_5z_6z_7 + z_1z_6z_7 = 0 \\
z_1z_2z_6z_7 + z_1z_5z_6z_7 = 0 \\
z_1z_2z_3z_4z_5z_6 + z_1z_2z_5z_6z_7 + z_1z_4z_5z_6z_7 = 0 \\
z_1z_2z_3z_4z_5z_6z_7 + z_1z_2z_4z_5z_6z_7 = 0 \\
z_1z_2z_3z_4z_5z_6z_7 - 1 = 0 \\
\end{cases}
\]  
(26)

By construction of \( M \), observe that all polynomials have the same power of \( z_0 \), so \( z_0 \) can be factored out. Removing \( z_0 \) from the initial form system, we find a solution

\[
z_0 = t, \ z_1 = -I, \ z_2 = -\frac{1}{2} - \frac{I}{2}, \ z_3 = -1, \ z_4 = 1 + I, \ z_5 = \frac{1}{2} + \frac{I}{2}, \ z_6 = I, \ z_7 = -1 - I
\]  
(27)

where \( I = \sqrt{-1} \). This solution is a regular solution. We set \( z_0 = t \), where \( t \) is the variable for the Puiseux series. In the computation of the second term, we assume the Puiseux series of the form

\[
\begin{align*}
z_0 &= t \\
z_1 &= -I + c_1t \\
z_2 &= \frac{1}{2} - \frac{I}{2} + c_2t \\
z_3 &= -1 + c_3t \\
z_4 &= 1 + I + c_4t \\
z_5 &= \frac{1}{2} + \frac{I}{2} + c_5t \\
z_6 &= I + c_6t \\
z_7 &= (-1 - I) + c_7t
\end{align*}
\]  
(28)

We first transform the cyclic 8-roots system \( C_8(x) = 0 \) using the coordinate transformation given by (25) and then substitute the assumed series form into this new system. Since the next term in the series is of form \( c_t t^l \), we collect all the coefficients of \( t^l \) and solve the linear system of equations. The second term in the Puiseux series expansion for the cyclic 8-root system, in the particular direction \( v \), has the form

\[
\begin{align*}
z_0 &= t \\
z_1 &= -I + (-1 - I)t \\
z_2 &= \frac{1}{2} - \frac{I}{2} + \frac{1}{2}t \\
z_3 &= -1 \\
z_4 &= 1 + I - t \\
z_5 &= \frac{1}{2} + \frac{I}{2} - \frac{1}{2}t \\
z_6 &= I + (1 + I)t \\
z_7 &= (-1 - I) + t
\end{align*}
\]  
(29)
Because of the regularity of the solution of the initial form system and the second term of the Puiseux series, we have a symbolic-numeric representation of a quadratic solution curve.

If we place the same pretropism in another row in the unimodular matrix, then we can develop the same curve starting at a different coordinate plane. This move is useful if the solution curve would not be in general position with respect to the first coordinate plane. For symmetric polynomial systems, we apply the permutations to the pretropism, the initial form systems, and its solutions to find Puiseux series for different solution curves, related to the generating pretropism by symmetry.

For the pretropism \( v = (1, -1, 1, -1, 1, -1, 1) \), the initial form is

\[
\text{in}_v(C_8)(x) = \begin{cases} 
  x_1 + x_3 + x_5 + x_7 = 0 \\
  x_0x_1 + x_0x_7 + x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_6 + x_6x_7 = 0 \\
  x_0x_1x_7 + x_1x_2x_3 + x_2x_4x_5 + x_3x_4x_5 = 0 \\
  x_0x_1x_2x_3 + x_0x_1x_2x_7 + x_0x_1x_6x_7 + x_0x_5x_6x_7 \\
  + x_1x_2x_3x_4 + x_2x_3x_4x_5 + x_3x_4x_5x_6 + x_4x_5x_6x_7 = 0 \\
  x_0x_1x_2x_4x_5 + x_0x_1x_2x_5x_6x_7 + x_0x_1x_2x_3x_4x_5x_6x_7 \\
  + x_0x_1x_4x_5x_6x_7 + x_0x_3x_4x_5x_6x_7 + x_1x_2x_3x_4x_5x_6x_7 = 0 \\
  x_0x_1x_2x_3x_4x_5x_6x_7 + x_0x_1x_2x_3x_5x_6x_7 + x_1x_2x_3x_4x_5x_6x_7 = 0 \\
  x_0x_1x_2x_3x_4x_5x_6x_7 - 1 = 0
\end{cases}
\]

The coordinate transformation for the initial form system (30) is given by the unimodular matrix \( M \) equal to the identity matrix, except for its first row which is the tropism \((1, -1, 1, -1, 1, -1, 1, -1)\).

The coordinate transformation \( x = z^M \) yields \( x_0 = z_0, x_1 = z_1/z_0, x_2 = z_2/z_0, x_3 = z_3/z_0, x_4 = z_0z_4, x_5 = z_5/z_0, x_6 = z_0z_6, x_7 = z_7/z_0 \). Applying the coordinate transformation to (30) gives

\[
\text{in}_v(C_8)(x = z^M) = \begin{cases} 
  z_1 + z_3 + z_5 + z_7 = 0 \\
  z_1z_2 + z_2z_3 + z_3z_4 + z_4z_5 + z_5z_6 + z_6z_7 + z_1 + z_7 = 0 \\
  z_1z_2z_3 + z_2z_3z_4 + z_3z_4z_5 + z_4z_5z_6 + z_5z_6z_7 + z_1z_2z_3 + z_2z_3z_4 = 0 \\
  z_1z_2z_3z_4 + z_2z_3z_4z_5 + z_3z_4z_5z_6 + z_4z_5z_6z_7 + z_1z_2z_3z_4 = 0 \\
  z_1z_2z_3z_4z_5 + z_2z_3z_4z_5z_6 + z_3z_4z_5z_6z_7 + z_1z_2z_3z_4z_5 + z_2z_3z_4 = 0 \\
  z_1z_2z_3z_4z_5z_6 + z_2z_3z_4z_5z_6z_7 + z_3z_4z_5z_6z_7 = 0 \\
  z_1z_2z_3z_4z_5z_6z_7 - 1 = 0
\end{cases}
\]  

The initial form system (31) has 72 solutions. Among the 72 solutions, a solution of the form

\[
z_0 = t, \ z_1 = -1, \ z_2 = I, \ z_3 = -I, \ z_4 = -1, \ z_5 = 1, \ z_6 = -I, \ z_7 = I,
\]

here expressed in the original coordinates,

\[
x_0 = t, \ x_1 = -1/t, \ x_2 = It, \ x_3 = -I/t, \ x_4 = -t, \ x_5 = 1/t, \ x_6 = -It, \ x_7 = I/t
\]
satisfies the cyclic 8-roots entirely. Applying the cyclic permutation of this solution set we can obtain the remaining 7 solution sets, which also satisfy the cyclic 8-roots system.

In [53], a formula for the degree of the curve was derived, based on the coordinates of the tropism and the number of initial roots for the same tropism. We apply the formula of [53] to the Puiseux series developments for the cyclic 8-roots problem and obtain 144 as the known degree of the space curve of the one dimensional solution set.

\[
(1, -1, 1, -1, 1, -1, 1, -1) \\
(1, -1, 0, 1, 0, 0, -1, 0) \rightarrow (1, 0, 0, -1, 0, 1, -1, 0) \\
(1, 0, -1, 0, 0, 1, 0, -1) \rightarrow (1, 0, -1, 1, 0, -1, 0, 0) \\
(1, 0, -1, 1, 0, -1, 0, 0) \rightarrow (1, 0, -1, 0, 0, 1, 0, -1) \\
(1, 0, 0, -1, 0, 1, -1, 0) \rightarrow (1, -1, 0, 0, 1, 0, -1, 0) \\
8 \times 2 = 16 \\
8 \times 2 + 8 \times 2 = 32 \\
8 \times 2 + 8 \times 2 = 32 \\
8 \times 2 + 8 \times 2 = 32 \\
TOTAL = 144
\]

Table 3: Tropisms, cyclic permutations, and degrees for the cyclic 8 solution curve.

Using the same polyhedral method we can find all the isolated solutions of the cyclic 8-roots system. We conclude this subsection with some empirical observations on the time complexity. In the direction of tropism \((1, -1, 0, 1, 0, 0, -1, 0)\), there exists a second term in the Puiseux series. The procedure solves the initial form system, which yields 40 solutions, and checks whether the first term satisfies the cyclic 8-roots system. Then it proceeds to construct and compute the second term in the Puiseux series. The total time required is 35.5 seconds, which includes 28 milliseconds that PHCpack needed to solve the initial form system. For the tropism \((1, -1, 1, -1, 1, -1, 1, -1)\) there is no second term in the Puiseux series. In this direction, the first term solves the entire cyclic 8-roots system. Hence, the procedure for construction and computation of the second term does not run. It takes PHCpack 12 seconds to solve the initial form system, whose solution set consists of 509 solutions. Determining that there is no second term for the 509 solutions, takes 199 seconds. Given their numbers of solutions, the ratio for time comparison is given by \(\frac{509}{35} \approx 14.54\). However, given that for tropisms \((1, -1, 0, 1, 0, 0, -1, 0)\) the procedure for construction and computation of the second term does run, unlike for tropism \((1, -1, 1, -1, 1, -1, 1, -1)\), the ratio for time comparison is not precise enough. A more accurate ratio for comparison is \(\frac{199}{35} \approx 5.686\).

5.3 Cyclic 12-roots

The generating solutions to the quadratic space curve solutions of the cyclic 12-roots problem are in Table 4. As the result in the Table 4 is given in the transformed coordinates, we return the solutions to the original coordinates. For any solution generator \((r_1, r_2, \ldots, r_{11})\) in Table 4:

\[
\begin{align*}
&z_0 = t, \quad z_1 = r_1, \quad z_2 = r_2, \quad z_3 = r_3, \quad z_4 = r_4, \quad z_5 = r_5, \\
&z_6 = r_6, \quad z_7 = r_7, \quad z_8 = r_8, \quad z_9 = r_9, \quad z_{10} = r_{10}, \quad z_{11} = r_{11}
\end{align*}
\]  

(34)

and turning to the original coordinates we obtain

\[
\begin{align*}
x_0 &= t, \quad x_1 = r_1/t, \quad x_2 = r_2t, \quad x_3 = r_3/t, \quad x_4 = r_4t, \quad x_5 = r_5/t \\
x_6 &= r_6t, \quad x_7 = r_7/t, \quad x_8 = r_8t, \quad x_9 = r_9/t, \quad x_{10} = r_{10}t, \quad x_{11} = r_{11}/t
\end{align*}
\]  

(35)
Application of the degree formula of [53] shows that all space curves are quadrics. Compared to [43], we arrive at this result without the application of any factorization methods.

6 Concluding Remarks

Inspired by an effective proof of the fundamental theorem of tropical algebraic geometry, we outlined in this paper a polyhedral method to compute Puiseux series expansions for solution curves of polynomial systems. The main advantage of the new approach is the capability to exploit permutation symmetry. For our experiments, we relied on cddlib and Gfan for the pretropisms, the blackbox solver of PHCpack for solving the initial form systems, and Sage for the manipulations of the Puiseux series.

References


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Table 4: Generators of the roots of the initial form system \( x \in (C_2 \langle 1 \rangle) \) and \( x = 0 \) with the tropism \( \textbf{V} = (1, -1, 1, -1, 1, -1, 1, -1, -1, 1, -1, 1) \) in the transformed \( z \) coordinates. Every solution defines a solution curve of the cyclic 12-roots system.