Outline

MCS 260 Lecture 12
Introduction to Computer Science
Jan Verschelde, 8 February 2010

Simulation

Monte Carlo methods

- In a mathematical model with uncertainties, events occur with assigned probabilities.
- Simulation consists in the repeated drawing of samples according to a probability distribution.
  We count the number of successful samples.
- The Law of Large Numbers states that the arithmetic average of the observed successes converges to the expected value or mean of the experiment, as the number of experiments increases.
- Monte Carlo methods are listed among the Top Ten Algorithms of the 20th century.

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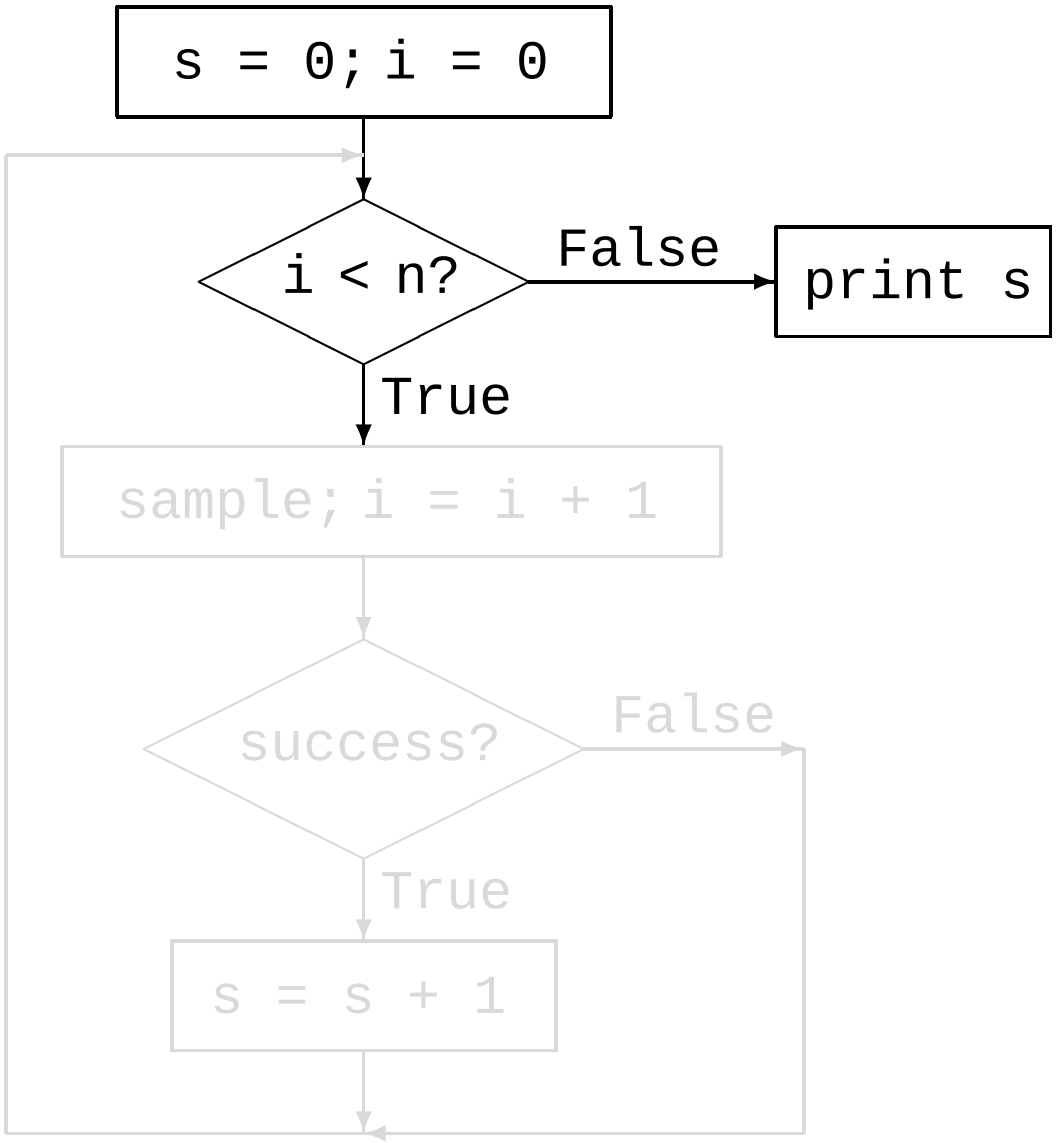
repeat until: break

Flowchart for Simulations



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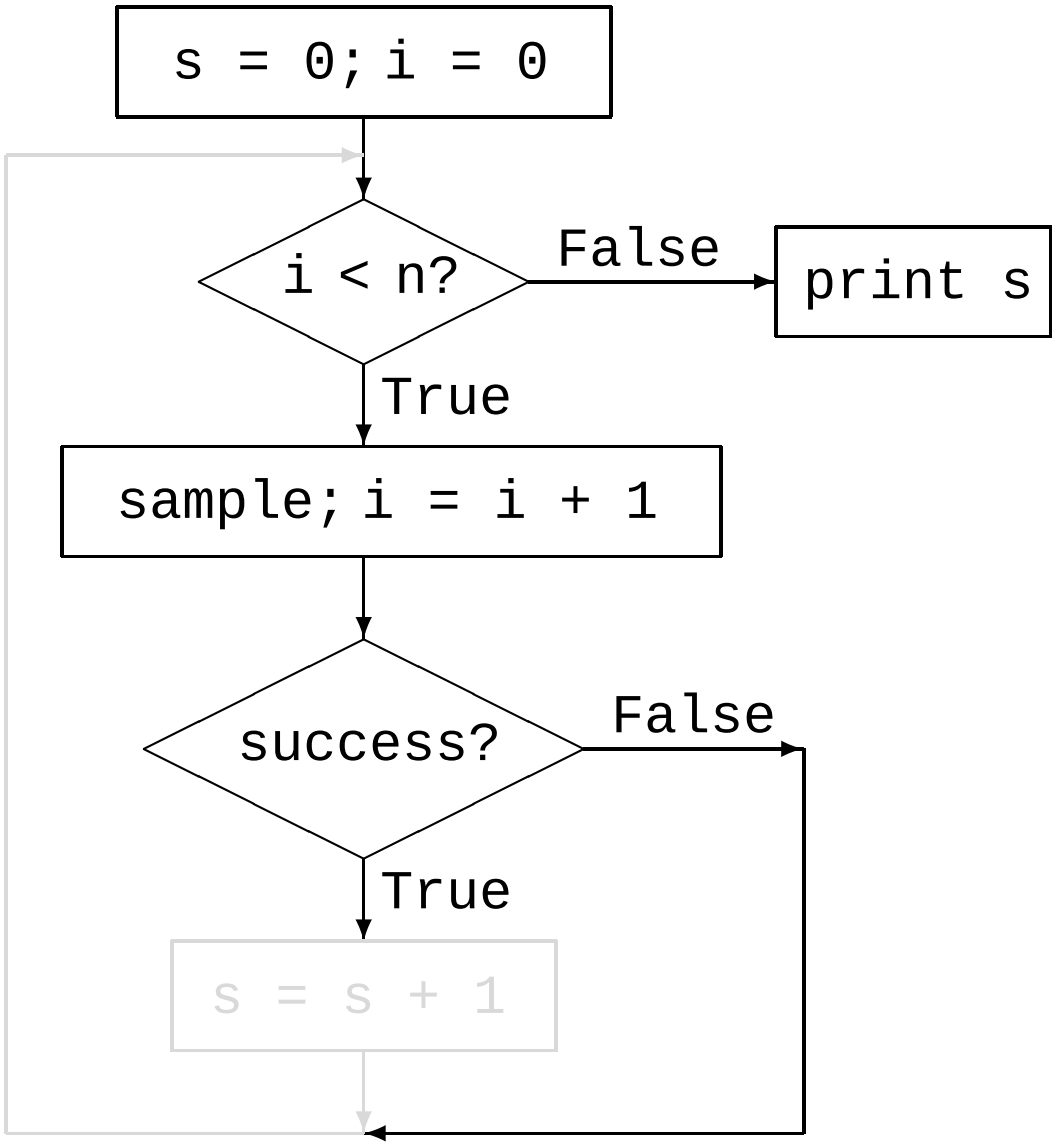

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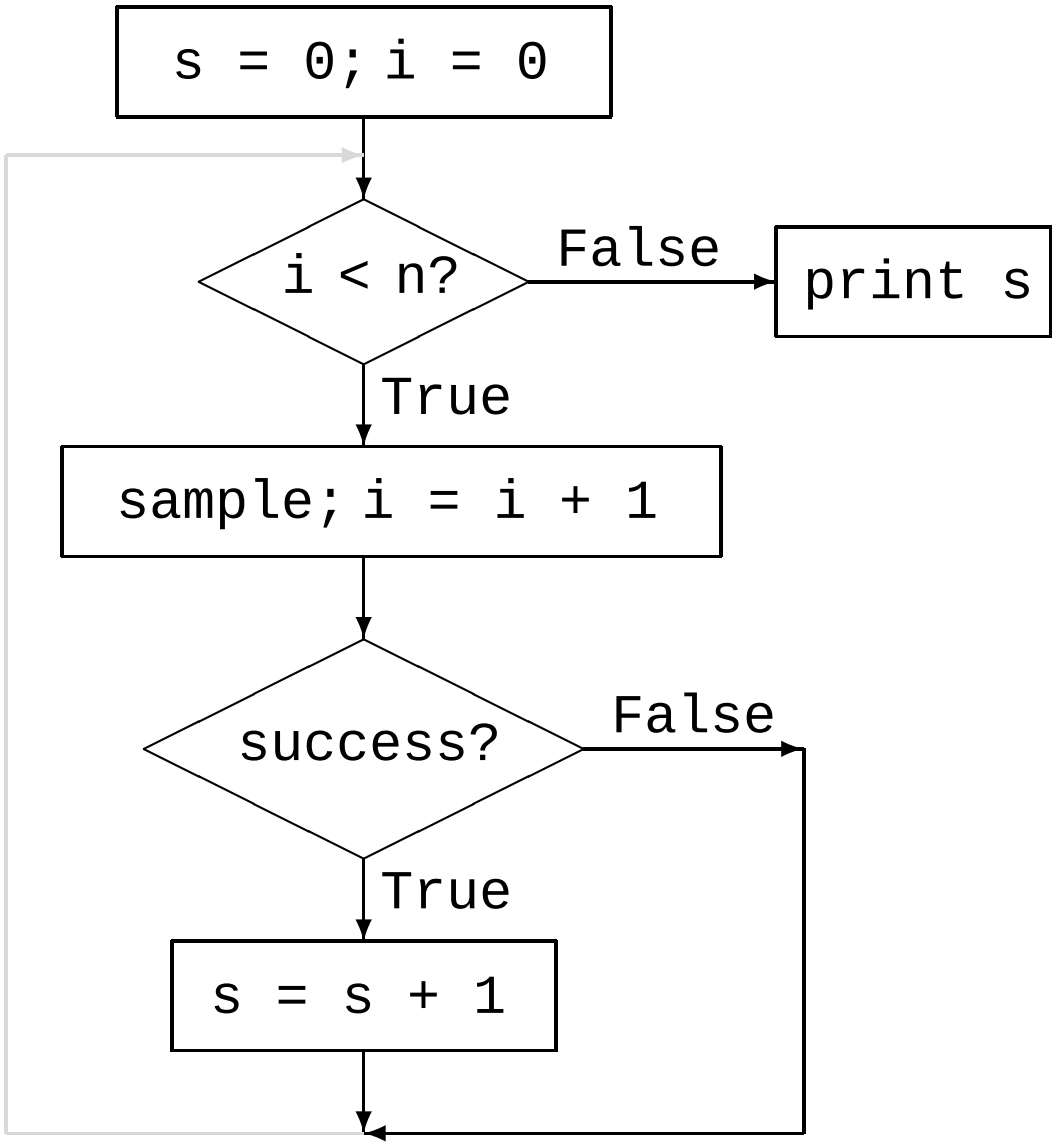

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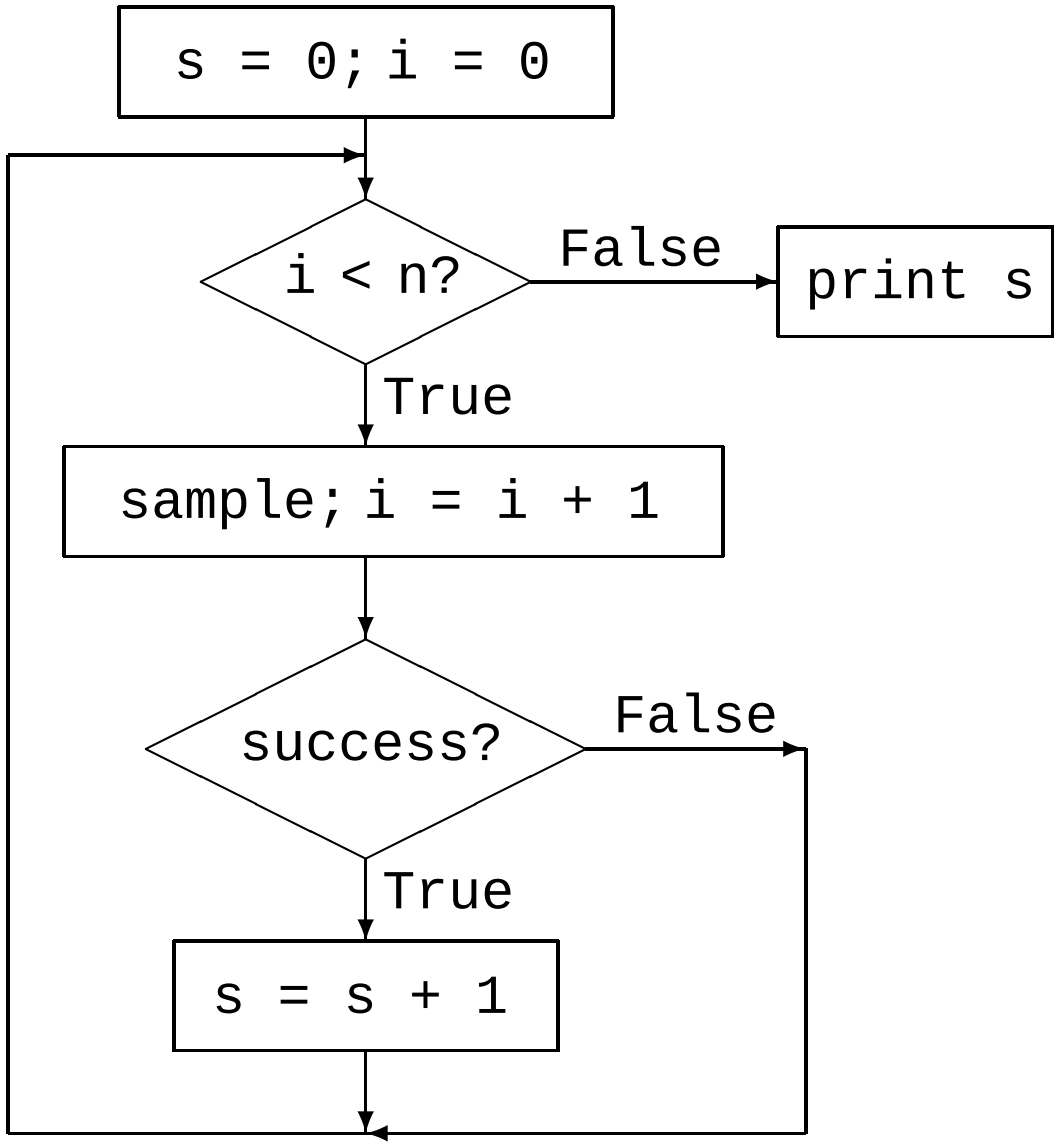

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Random Numbers

as available in Python

Random number generators are in the module `random`.

Three things we need to know:

1. `import random` loads the module into a session. Afterwards, `help(random)` shows a description of the definitions and functions offered by the module.
2. `random.seed()`
   Giving a fixed number as argument results in the same sequence of random numbers.
3. `r = random.uniform(a,b)`
   `r` is a randomly generated number, drawn from a uniform distribution over the interval `[a,b)`.

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Using Random Numbers

a sample program

A sample program randuse.py:

```
# L-12 MCS 260 Mon 8 Feb 2010  random numbers
#
import random           # use module random
random.seed(21342342) # get same sequence
print 'uniformly distributed random numbers'
a = input('give lower bound : ')
b = input('give upper bound : ')
r = random.uniform(a,b) # generate a number
print 'a random number in [%.2f,%.2f] : %.15f' \
      % (a,b,r)
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Estimating Areas and Volumes

high dimensional integrals

- Expected values are expressed as integrals.
  When many parameters are involved, the integration is high dimensional and only estimation is possible.
- The area of the unit disk is π.

Generate random uniformly distributed points with coordinates $(x, y) \in [-1, +1] \times [-1, +1]$.

We count a success when $x^2 + y^2 \leq 1$.

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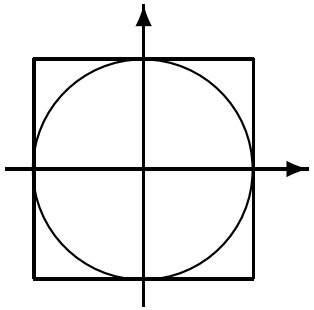

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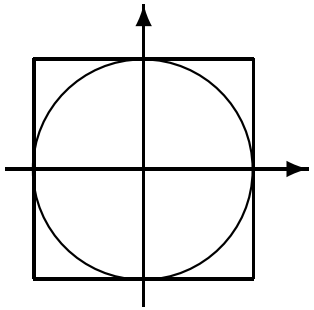

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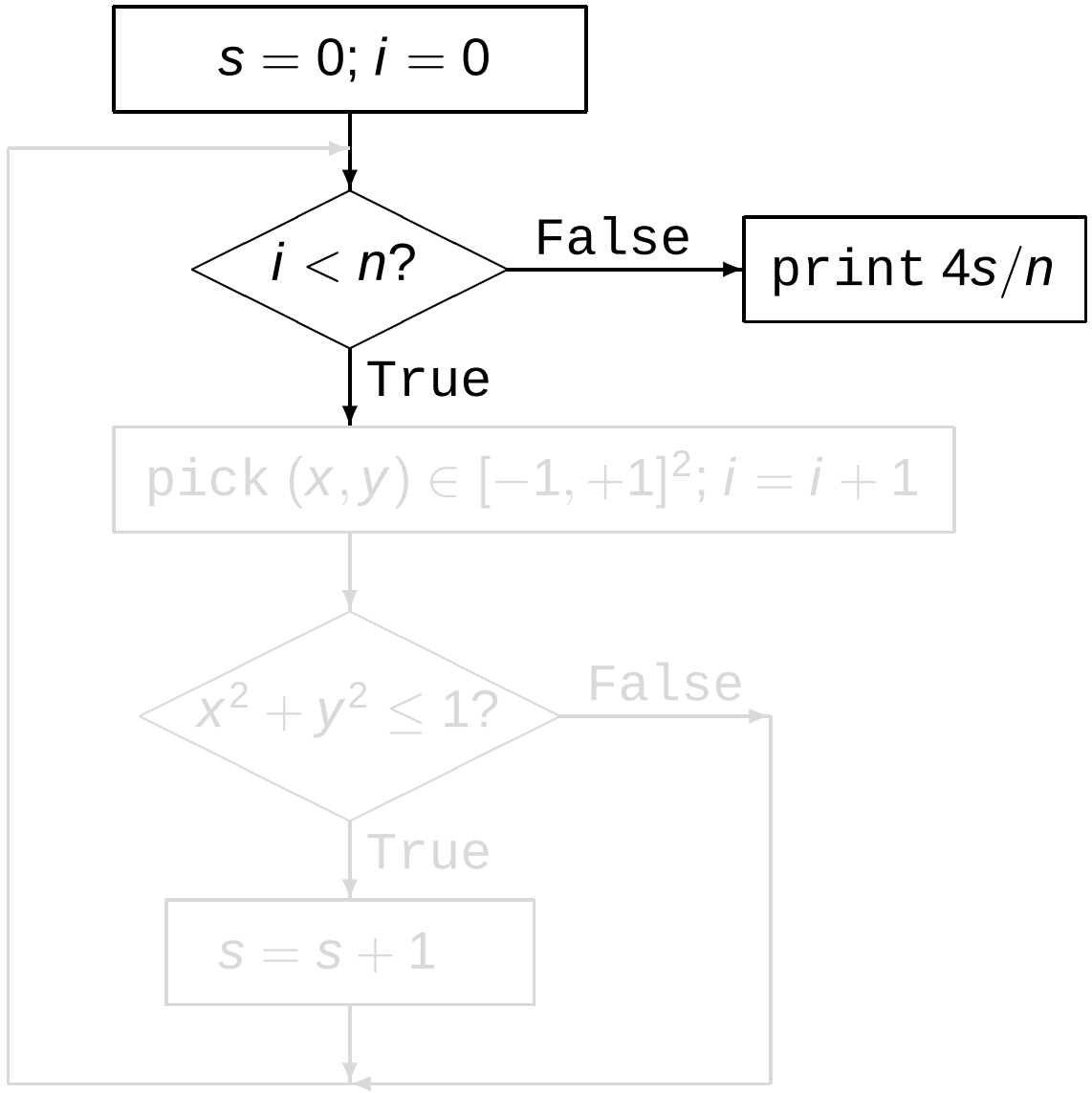

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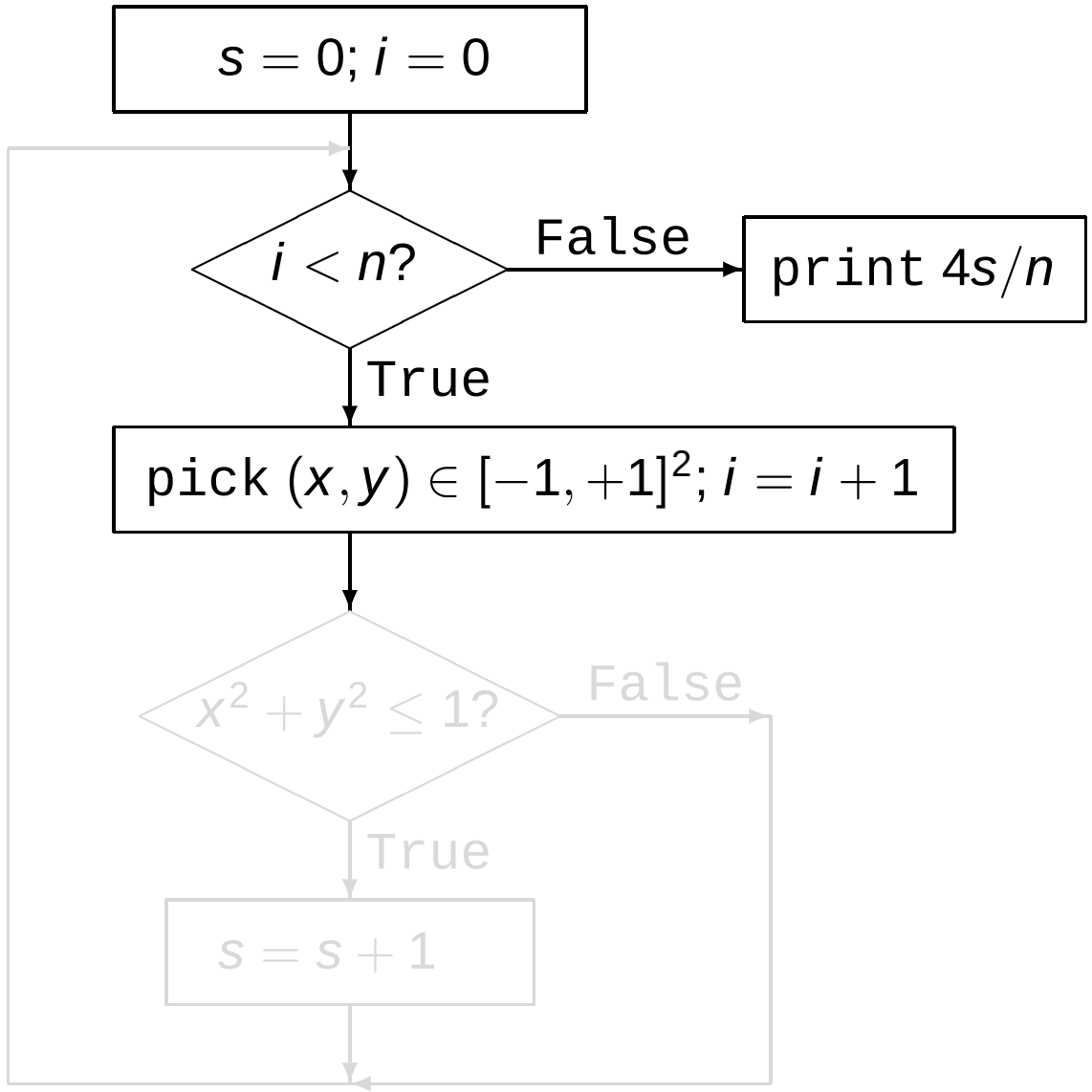

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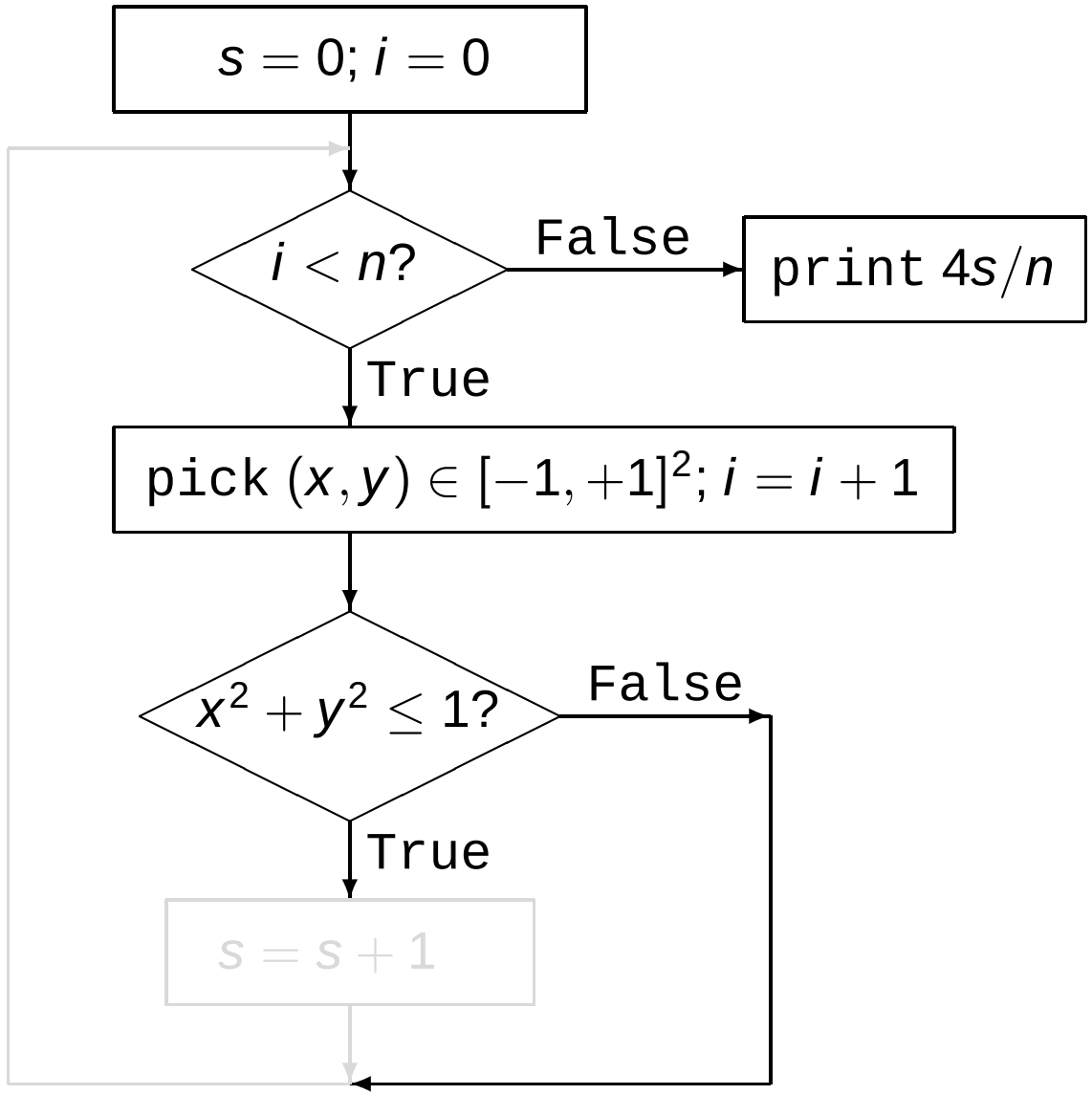

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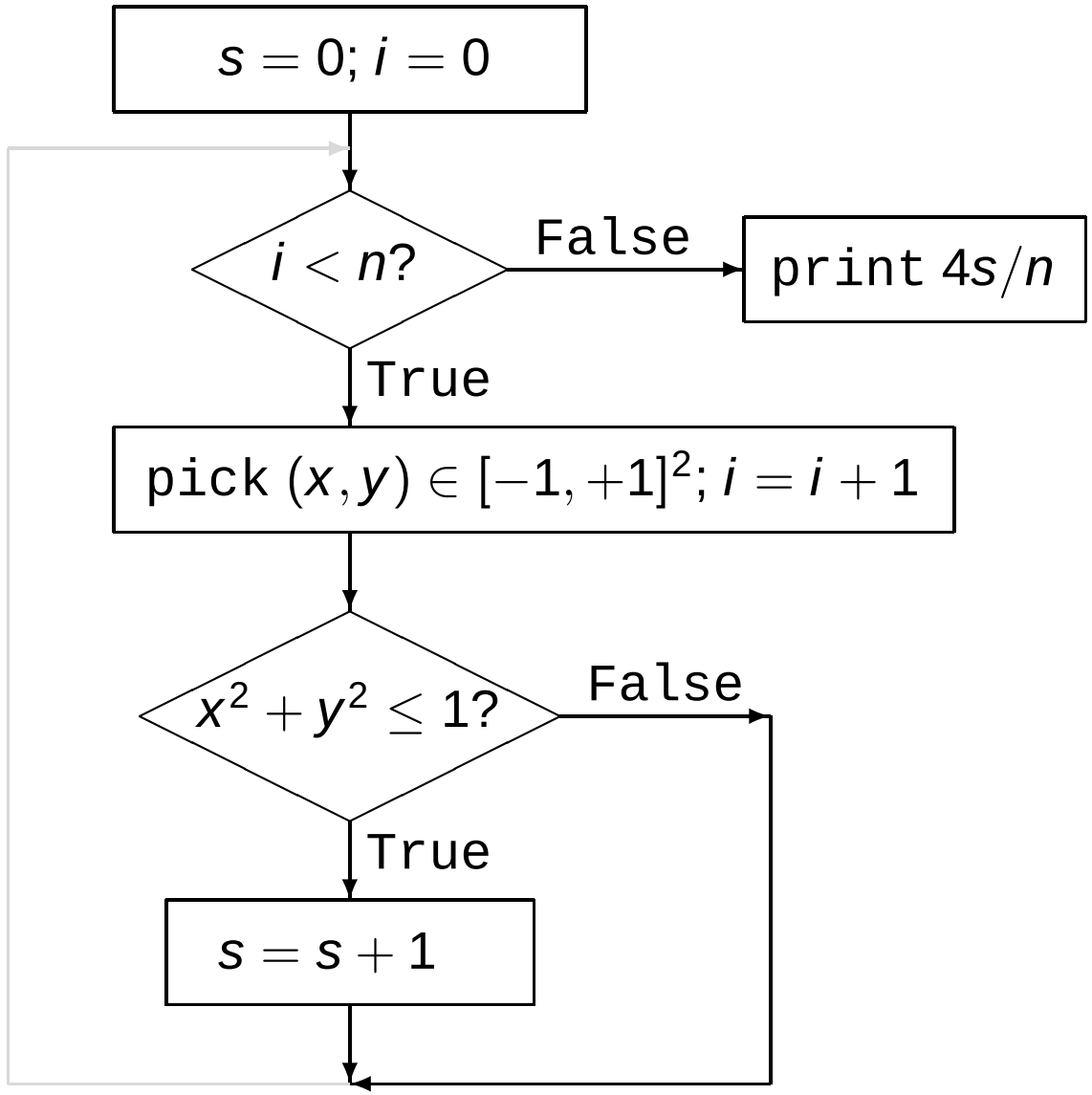

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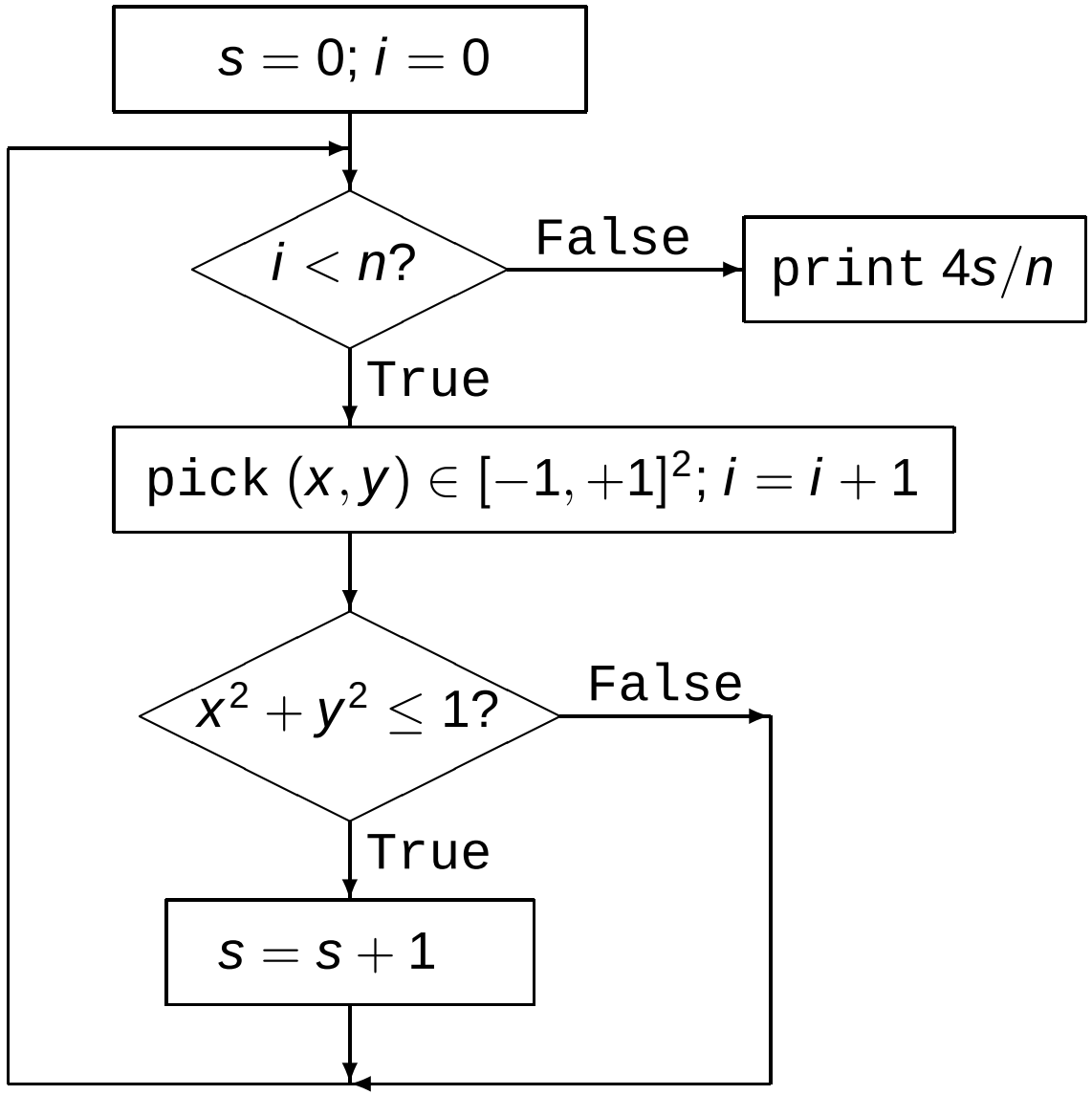

Estimating π

The program `mc4pi.py`:

```
# L-12 MCS 260 Mon 8 Feb 2010  Monte Carlo
#
import random
print 'Monte Carlo simulation for Pi'
n = input('Give number of runs : ')
s = 0
for i in range(0,n):
   x = random.uniform(-1,1)
   y = random.uniform(-1,1)
   if x**2 + y**2 <= 1: s += 1
print 'After %d runs : %f' % (n,4.0*s/n)
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Why multiply by 4? 4 is the area of $[-1,+1]^2$.

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Converting Numbers

Converting 123, from decimal into binary format:

n	$n/2$	$n \bmod 2$	
123	61	1	$123 = 61 \times 2 + 1$
61	30	1	$61 = 30 \times 2 + 1$
30	15	0	$30 = 15 \times 2 + 0$
15	7	1	$15 = 7 \times 2 + 1$
7	3	1	$7 = 3 \times 2 + 1$
3	1	1	$3 = 1 \times 2 + 1$
1	0	1	$1 = 0 \times 2 + 1$

$$\begin{aligned} 123 &= 1 + 2 \times 61 = 1 + 2 \times (1 + 2 \times 30) \\ &= 1 + 2 \times (1 + 2 \times (0 + 2 \times 15)) \\ &= 1 + 2 \times (1 + 2 \times (0 + 2 \times (1 + 2 \times 7))) \\ &= \ldots = 1111011 = 7B. \end{aligned}$$

The table shows the progression of the values of the variables in the loop, each row is one iteration.

Binary Expansions

repeat until loops

The bits of a number are the remainders of division by 2.

divmod() is an intrinsic operation:

```
>>> divmod(9,2)
(4, 1)
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Use as `(n,r) = divmod(n,2)`
to obtain remainder n%2 in `r` and to replace n by n/2.

Pseudo code to compute the binary expansion:

```
n = input()
repeat
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until (n == 0).
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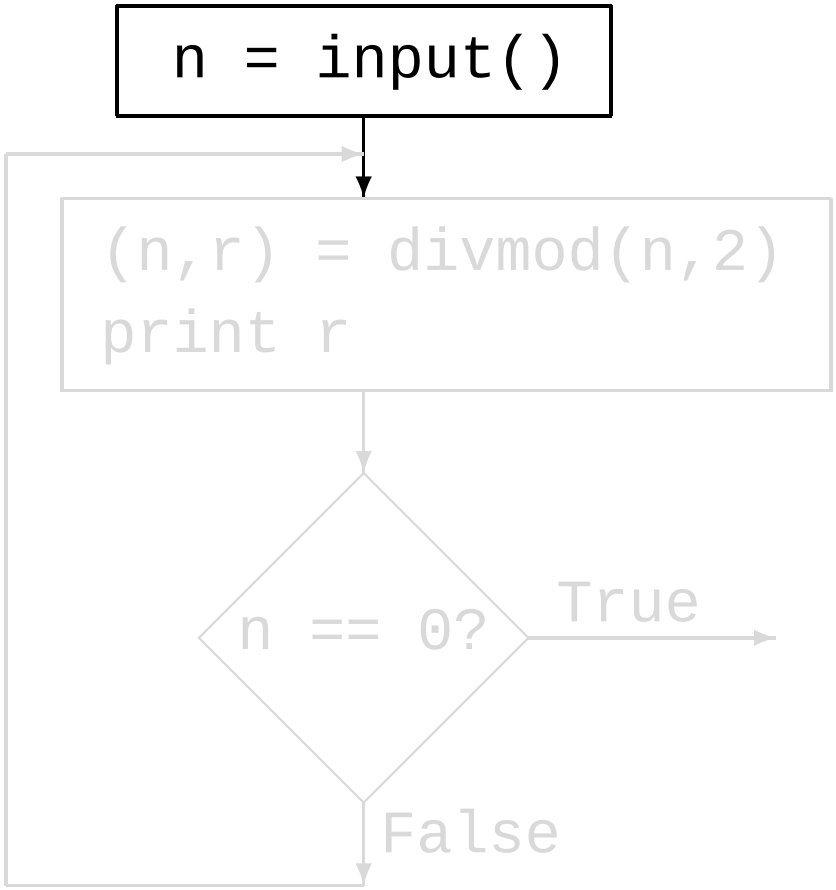

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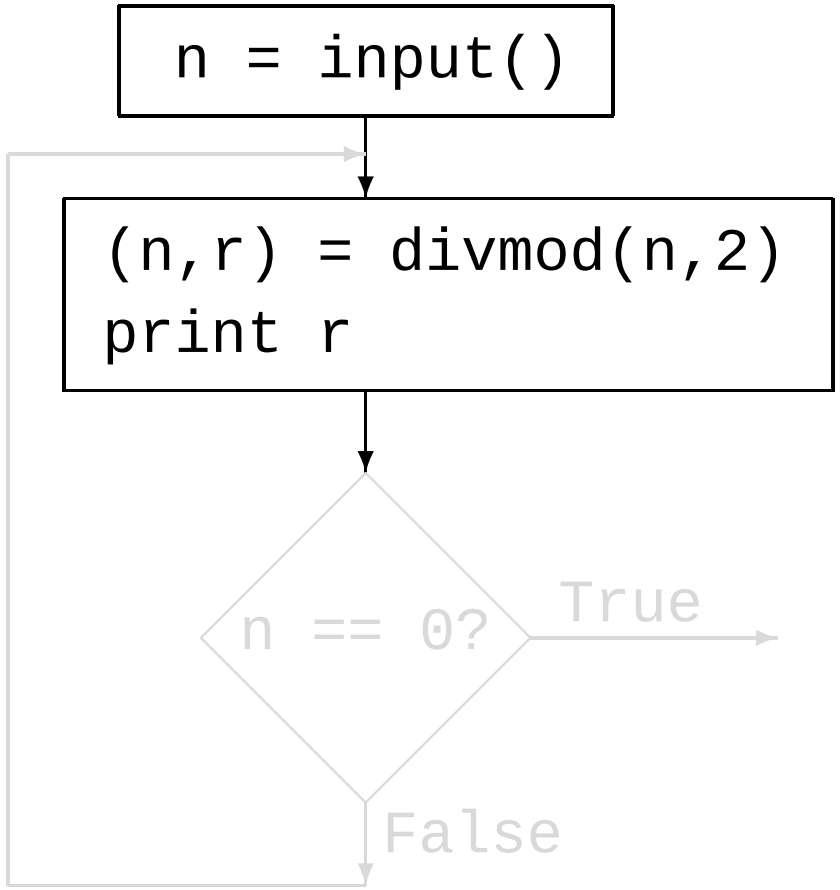

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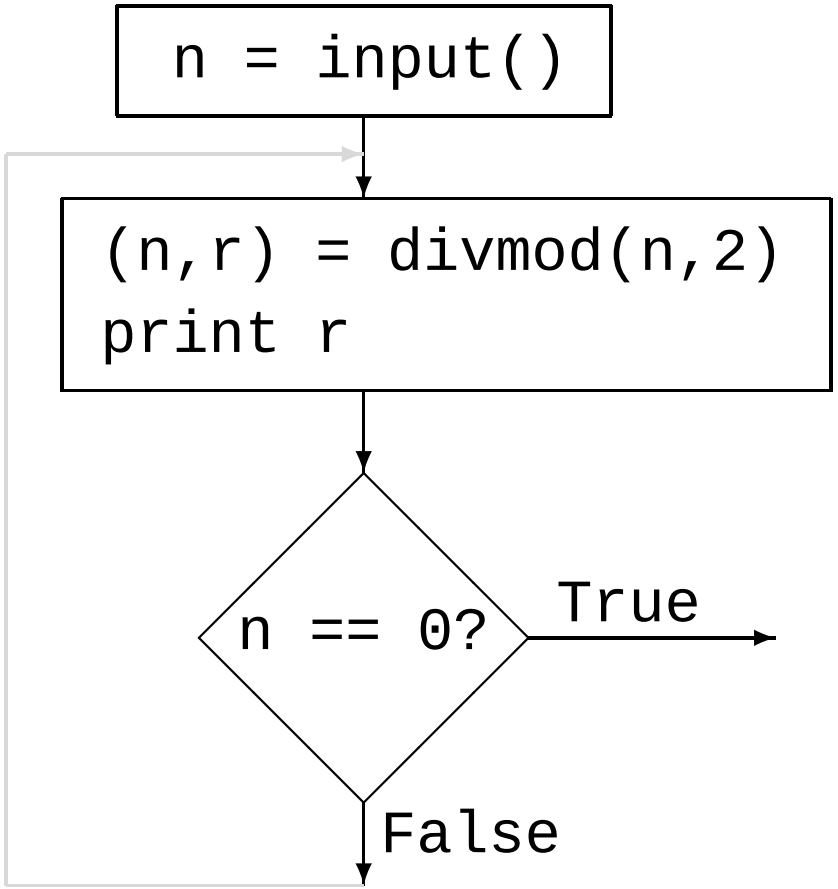

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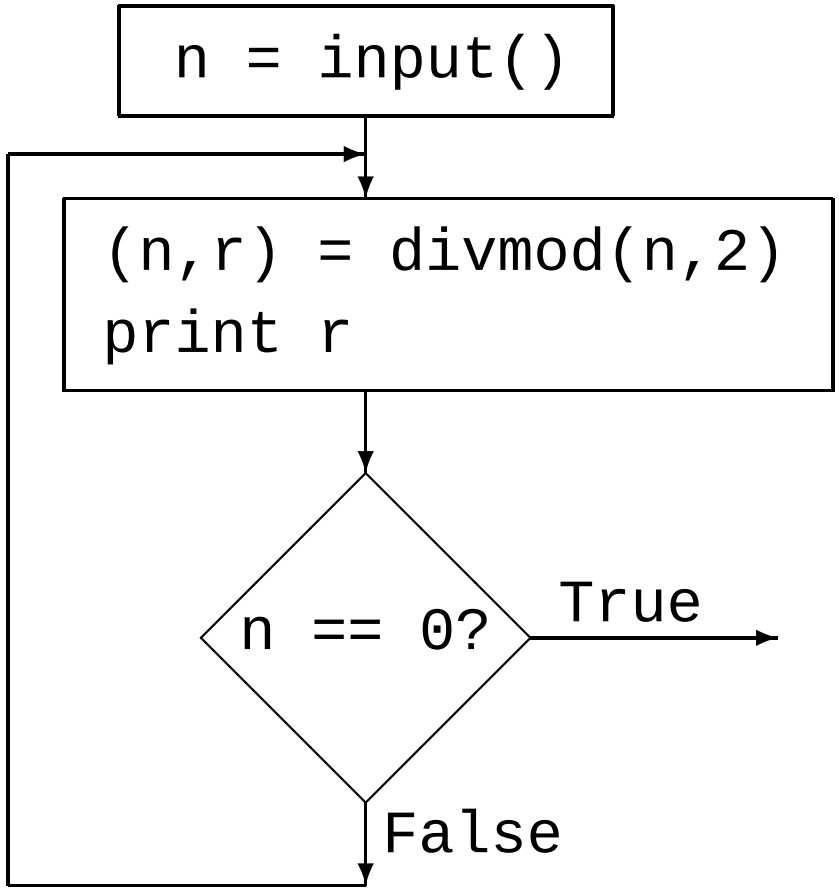

A first Python solution

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# L-12 MCS 260 Mon 8 Feb 2010  binary expansion
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# This first version prints the bits
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n = input('Give a number : ')
(n,r) = divmod(n,2)
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while n > 0:
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The break Statement

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To exit a loop inside the body of a loop, the statement `break` occurs usually within an `if` statement.

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The program below avoids the duplication of code:

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# Use of break for repeat until.
#
print 'computing the binary expansion'
n = input('Give a number : ')
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Exercise: how to print bits in correct order?

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Two Loops, Two Breaks

scope of a break

As long as the number typed in by the user is nonnegative, the loop continues.

```
# L-12 MCS 260 Mon 8 Feb 2010  binary expansion
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# A break only effects one loop.
#
print 'computing the binary expansion'
while True:
   n = input('Give a number (< 0 to exit) : ')
   if n < 0: break
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Arrays represent Matrices

using numpy

Python has no array type, we use `numpy`:

```
>>> from numpy import *
>>> A = zeros((2,3),int)
>>> A
array([[0, 0, 0],
       [0, 0, 0]])
```

We can define arrays via a list of lists:

```
>>> B = array([[1,2,3],[4,5,6]])
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Two ways of accessing elements:

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Random Arrays

A 2-by-3 array with elements in $[-10, +10]$:

```
>>> X = random.randint(-10,10,(2,3))
>>> X
array([[ -5, -10,   0],
       [  7,   1,  -5]])
```

Random arrays of floats:

```
>>> Y = random.rand(2,3)
>>> Y
array([[ 0.43049771,  0.62879832,  0.54863368],
       [ 0.37764022,  0.32186191,  0.43726834]])
>>> Z = random.randn(2,3)
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Running Simulations
repeat until: break

Search a 2-dimensional Array

Problem statement:

Input: A is an array of n rows and m columns, x is some number.

Output: if $A[i,j]$ equals x, then print (i,j), else print x does not occur in A.

We develop an interactive program:

1. The user provides n and m
2. The computer generates n-by-m array A of random integer numbers in the interval $[-100, +100]$.
3. The program prompts the user for x, and
4. searches A for x and prints search result.

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Output: if $A[i,j]$ equals x, then print (i,j), else print x does not occur in A.

We develop an interactive program:

1. The user provides n and m
2. The computer generates n-by-m array A of random integer numbers in the interval $[-100, +100]$.
3. The program prompts the user for x, and
4. searches A for x and prints search result.

Search a 2-dimensional Array

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Running the Code

at the command prompt

```
$ python findelem.py
give number of rows : 4
give number of columns : 5
a random 4-by-5 array :
[[-13  -9 -35 -82 -21]
 [-46  99  62  93  37]
 [ 88  47  66 -12  36]
 [-59  95  53 -97 -93]]

give a number : 47
found 47 at [2,1]
```

If the given number does not occur, then
`'%d does not occur in array'` is printed.

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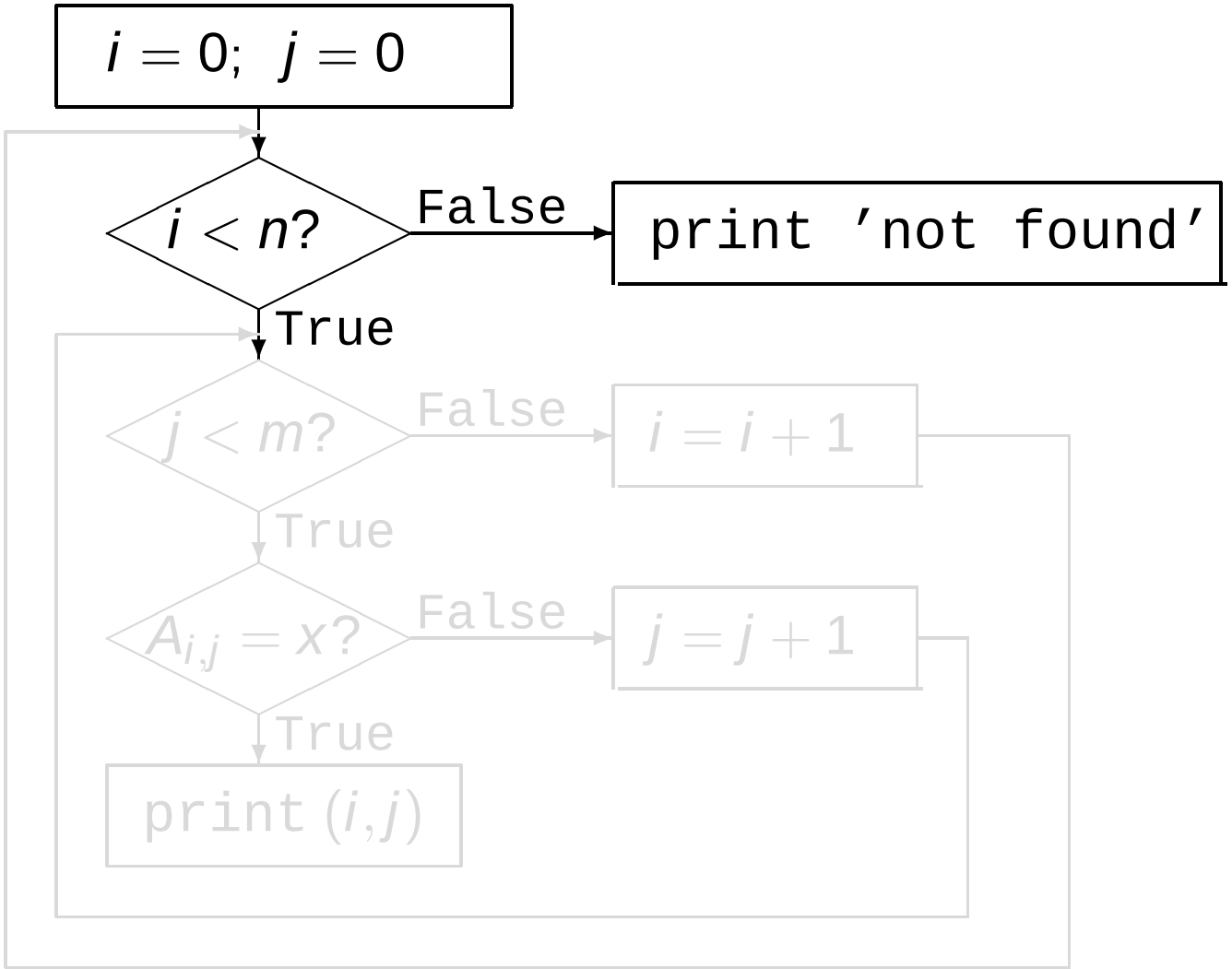
Flowchart



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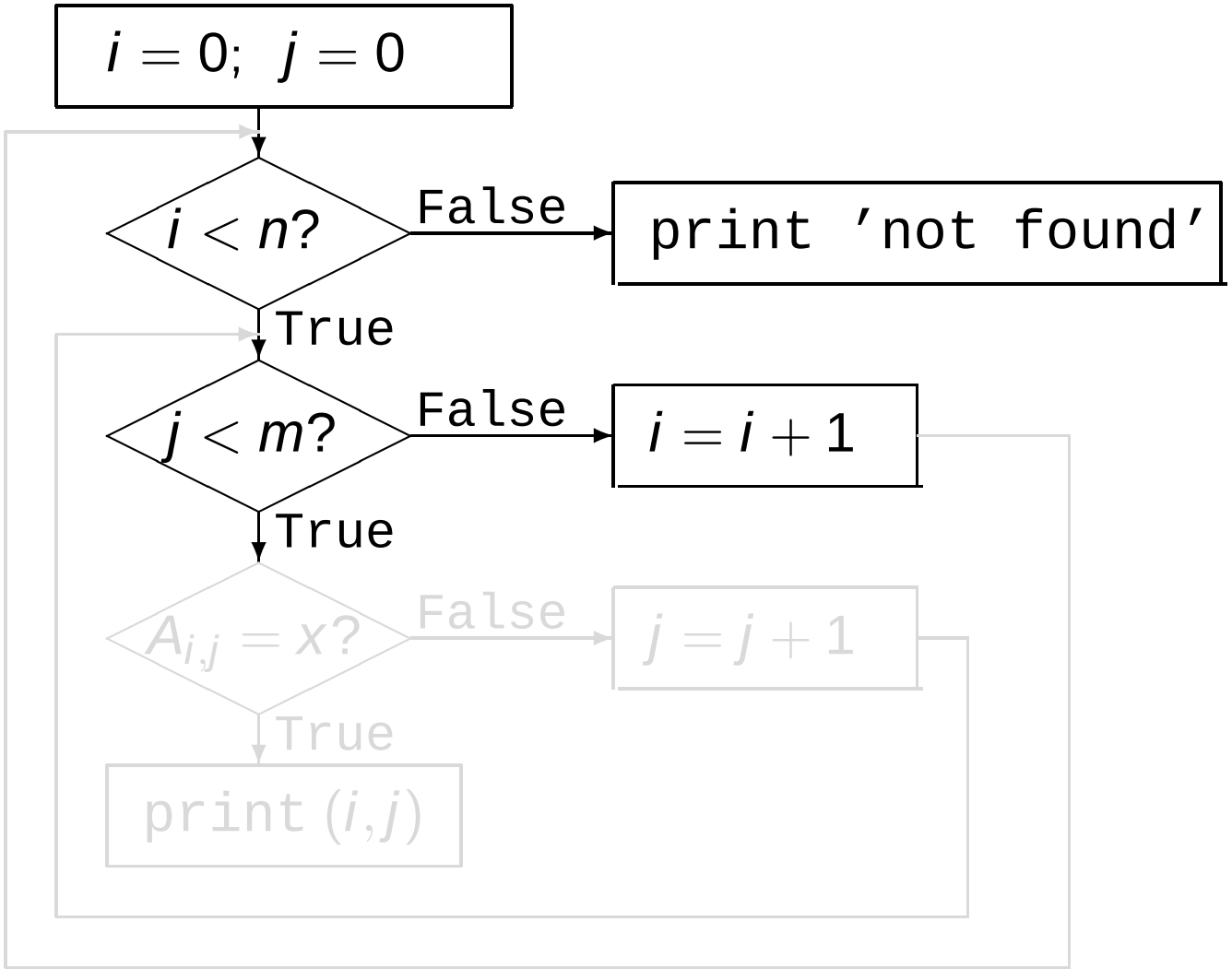

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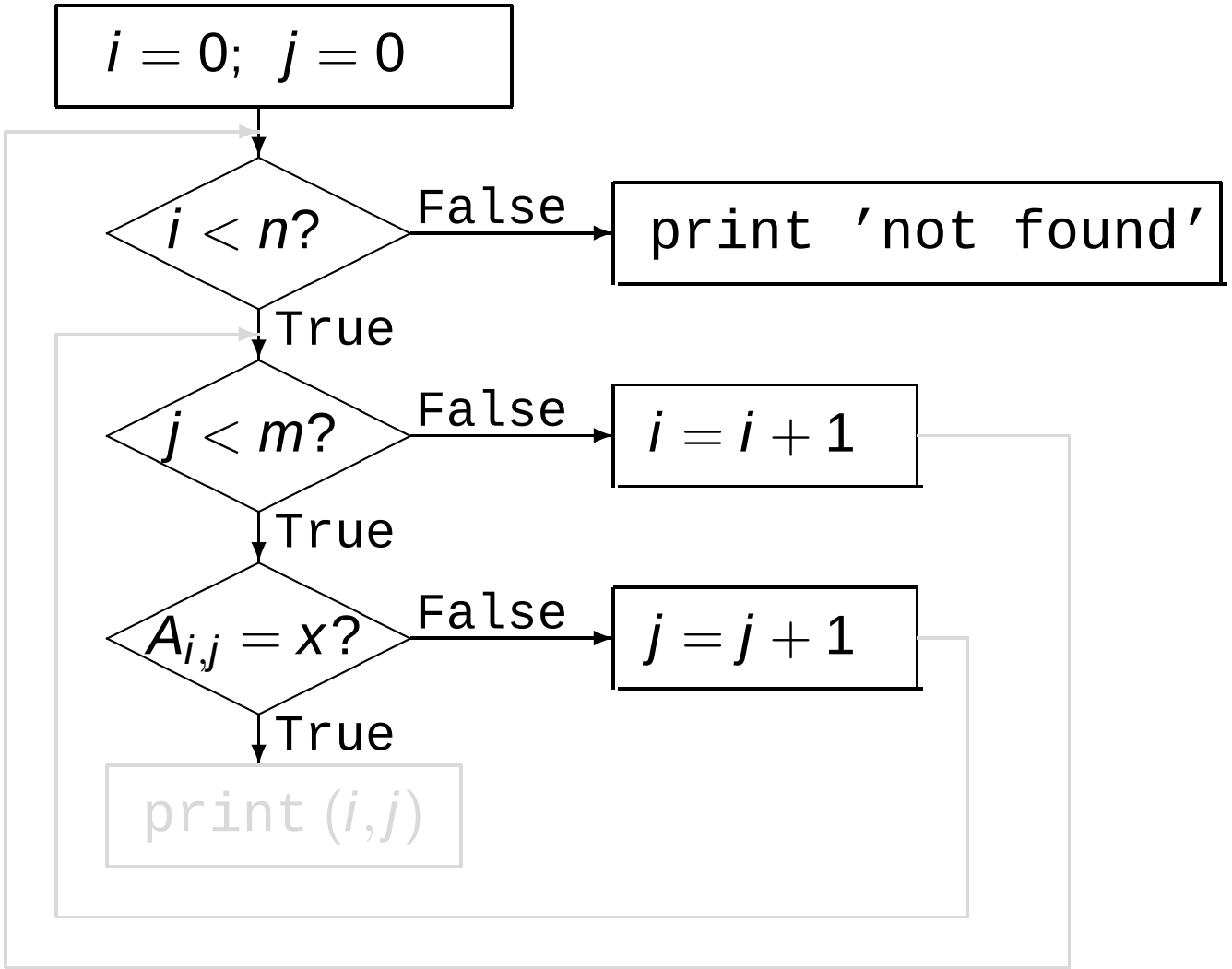

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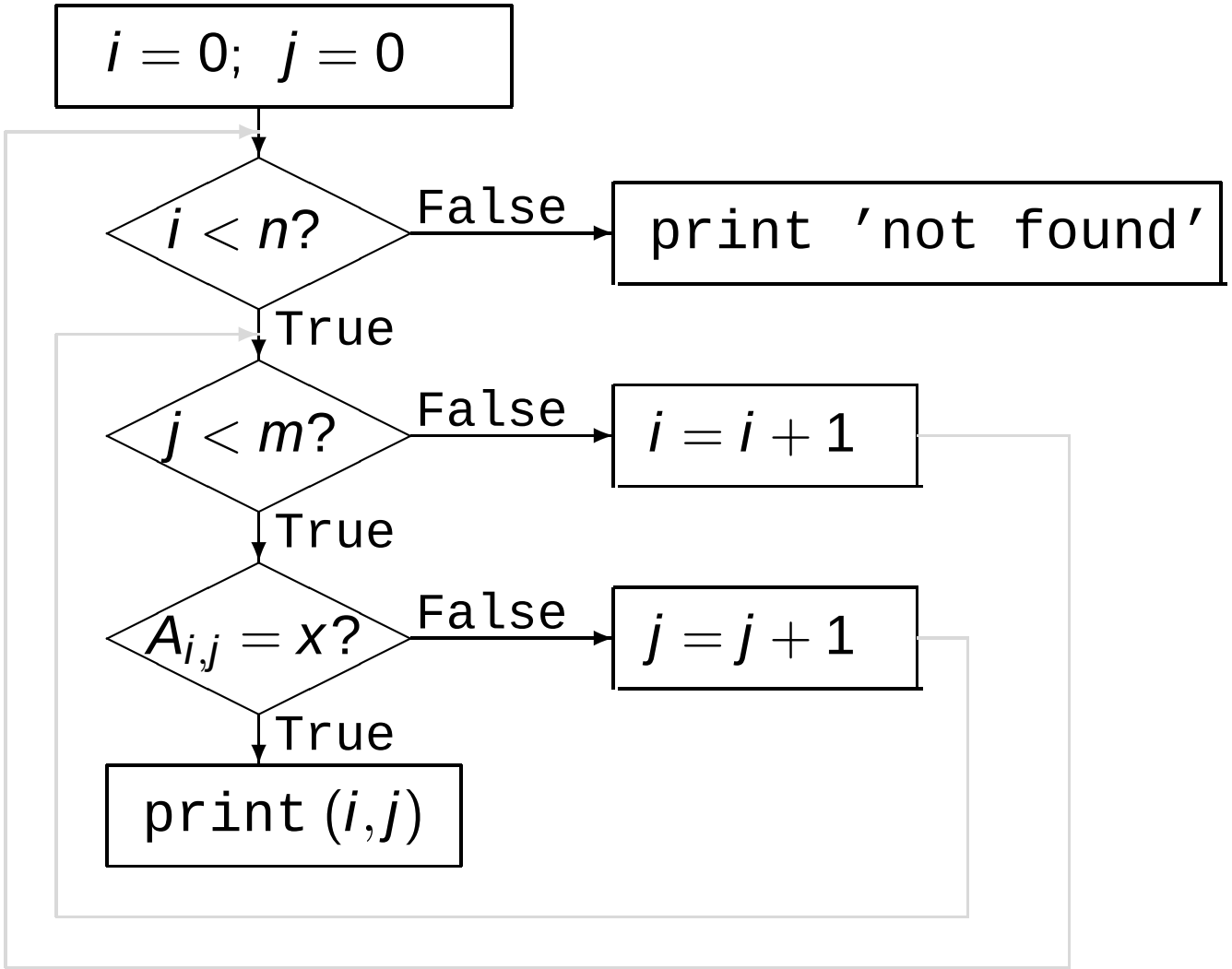

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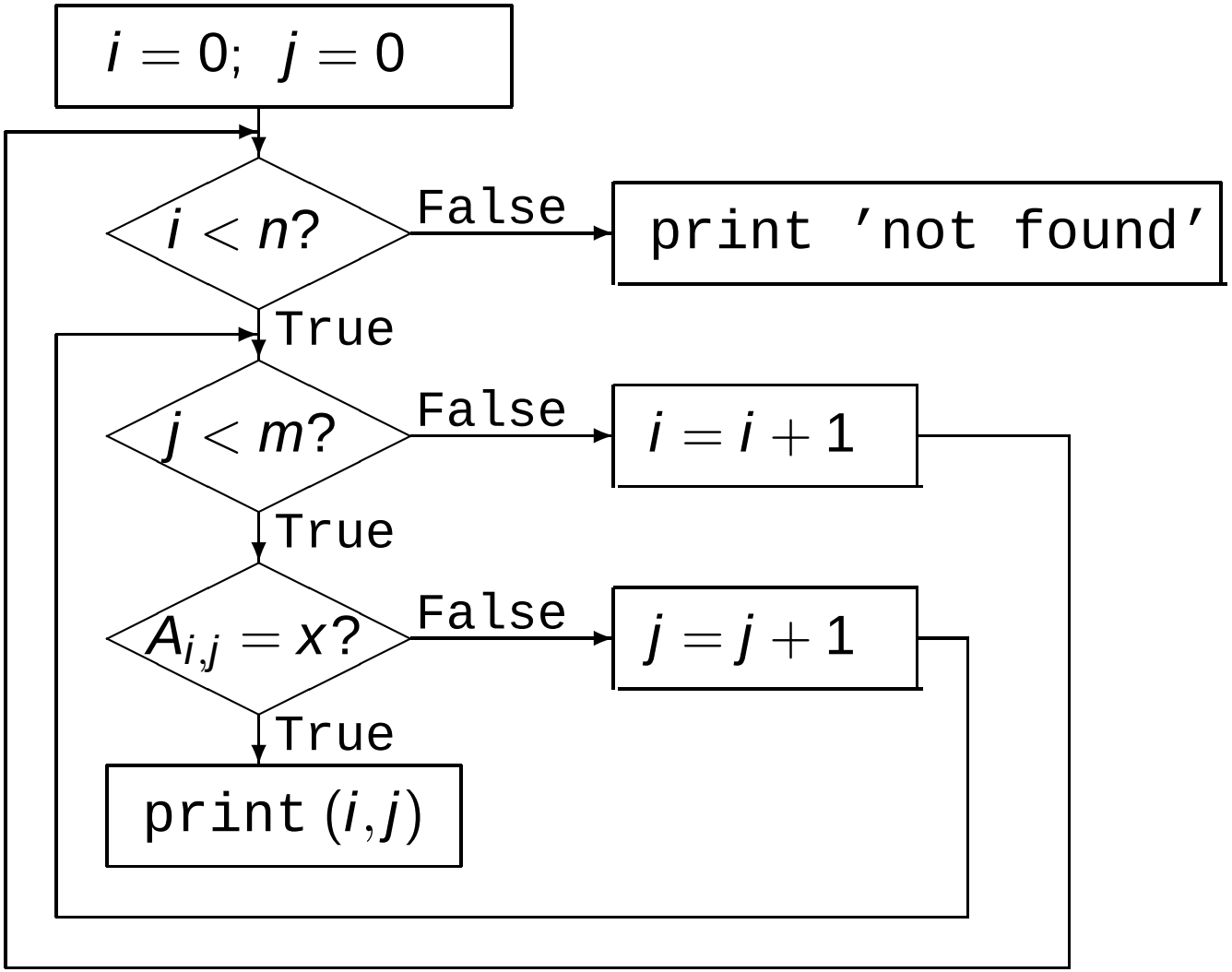

Start of findelem.py

```
# L-12 MCS 260 Mon 8 Feb 2010 : find element

# Illustration of a double for loop to find
# an element in a two dimensional array.

from numpy import *
n = input('give number of rows : ')
m = input('give number of columns : ')
A = random.randint(-100,100,(n,m))
print 'a random %d-by-%d array :\n' % (n,m), A
x = input('give a number : ')
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the double for loop

Search *n*-by-*m* array *A* for *x*:

```
found = False
for i in range(0,n):
   for j in range(0,m):
      if A[i,j] == x:
         found = True
         break
   if found: break
if found:
   print 'found %d at [%d,%d]' % (x,i,j)
else:
   print '%d does not occur in array' % x
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Assignments

1. Use a stack to store the bits in the binary expansion to print the bits *after* the loop in the correct order.
2. Given a list of numbers between 0 and 100, define the algorithm to assign a letter grade to each number:
   ≥ 90: A, $\in [80, 89]$: B, $\in [70, 79]$: C, etc.
   Report at the end how many As, Bs, Cs, etc.
   Write the algorithm in words and draw a flowchart.
3. Implement exercise 2 in Python.
4. Write a Python program that generates n numbers uniformly distributed in $[0, 1]$ and counts how many numbers are < 0.5.
5. Use turtle graphics to visualize the Monte Carlo method to estimate π. Represent the unit circle by a circle of radius equal to half of the width of the turtle window. Mark samples inside the disk by green circles of radius equal to 2 pixels, centered at the sample point. Use red circles for the points outside the disk.

Summary

We covered more of

- section 2.6 of *Python Programming in Context*,
- section 5.4 in *Computer Science, an overview*.