Generators, Recursion, and Fractals

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   - computing a list of Fibonacci numbers
   - defining a generator with `yield`
   - putting `yield` in the function `fib`
2. Recursive Functions
   - computing factorials recursively
   - computing factorials iteratively
3. Recursive Images
   - some examples
   - recursive definition of the Cantor set
   - recursive drawing algorithm

MCS 260 Lecture 41
Introduction to Computer Science
Jan Verschelde, 22 April 2016

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the Fibonacci numbers

The Fibonacci numbers are the sequence

$$0, 1, 1, 2, 3, 5, 8, \ldots$$

where the next number in the sequence is
the sum of the previous two numbers in the sequence.

Suppose we have a function:

```
def fib(k):
    """
    Computes the k-th Fibonacci number.
    """
```

and we want to use it to compute the first 10 Fibonacci numbers.

the function `fib`

```
def fib(k):
    """
    Computes the k-th Fibonacci number.
    """
    if k == 0:
        return 0
    elif k == 1:
        return 1
    else:
        (prevnum, nextnum) = (0, 1)
        for i in range(1, k):
            (prevnum, nextnum) = (nextnum, \
                prevnum + nextnum)
        return nextnum
```

the main program

```
def main():
    """
    Prompts the user for a number n and
    prints the first n Fibonacci numbers.
    """
    nbr = int(input('give a natural number n : '))
    fibnums = [fib(i) for i in range(nbr)]
    print(fibnums)
```

Running at the command prompt `$`

```
$ python fibnum.py
give a natural number n : 10
[0, 1, 1, 2, 3, 5, 8, 13, 21, 34]
```

Problem: with each call to `fib`, we recompute too much.

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defining a generator with `yield`

```
def counter(value=0):
    """
    Maintains a counter.
    """
    count = value
    while True:
        count = count + 1
        yield count
```

If saved in `show_yield.py`, then we can do

```
>>> from show_yield import counter
>>> mycounter = counter(3)
>>> next(mycounter)
4
>>> next(mycounter)
5
>>>
```

using a generator

```
def main():
    """
    Example of the use of a generator.
    """
    print('initializing counter ...')
    mycounter = counter(3)
    print('incrementing counter ...')
    print(next(mycounter))
    print('incrementing counter ...')
    print(next(mycounter))

if __name__ == "__main__":
    main()
```

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putting `yield` in the function `fib`

```
def fib(k):
    """
    Computes the k-th Fibonacci number, for k > 1
    """
    (prevnum, nextnum) = (0, 1)
    for i in range(k):
        (prevnum, nextnum) = (nextnum, prevnum + nextnum)
        yield nextnum
```

the main function

```
def main():
    """
    Prompts the user for a number n and
    prints the first n Fibonacci numbers.
    """
    nbr = int(input('give a natural number n : '))
    fibgen = fib(nbr)
    fibnums = [next(fibgen) for i in range(nbr)]
    print(fibnums)
```

Running at the command prompt:

```
$ python fibyield.py
give a natural number n : 10
[1, 2, 3, 5, 8, 13, 21, 34, 55, 89]
$
```

Observe the relabeling.

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Computing Factorials Recursively

rule based programming

Let n be a natural number.
By $n!$ we denote *the factorial* of n.

Its recursive definition is given by two rules:

1. for $n \leq 1$: $n! = 1$
2. if we know the value for $(n-1)!$
   then $n! = n \times (n-1)!$

Recursion is similar to mathematical proof by induction:

1. first we verify the trivial or base case
2. assuming the statement holds for all values
   smaller than n – the induction hypothesis –
   we extend the proof to n

the function `factorial`

```
def factorial(n):
    """
    computes the factorial of n recursively
    """
    if n <= 1:
        return 1
    else:
        return n*factorial(n-1)

def main():
    nbr = int(input('give a natural number n : '))
    fac = factorial(nbr)
    print('n! = ', fac)
    print('len(n!) = ', len(str(fac)))
```

tracing recursive functions

Calling `factorial` for $n = 5$:

```
factorial(5) call #0: call for n-1 = 4
 factorial(4) call #1: call for n-1 = 3
  factorial(3) call #2: call for n-1 = 2
   factorial(2) call #3: call for n-1 = 1
    factorial(1) call #4: base case, return 1
   factorial(2) call #3: returning 2
  factorial(3) call #2: returning 6
 factorial(4) call #1: returning 24
factorial(5) call #0: returning 120
```

Computes in returns:

return `1`, `1*2`, `1*2*3`, `1*2*3*4`, `1*2*3*4*5`

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running the recursive factorial

Long integers are no problem, but ...

```
$ python factorial.py
give a natural number n : 79
n! =  894618213078297528685144171539831652
069808216779571907213868063227837990693501
860533618108410101760000000000000000000
len(n!) =  117
```

Exploiting Python long integers:

```
$ python factorial.py
give a number : 1234
...
RuntimeError: maximum recursion depth exceeded
```

An exception handler will compute $n!$ iteratively.

stack of function calls

The execution of recursive functions
requires a stack of function calls.

For example, for `n = 5`, the stack grows like

```
    factorial(1) call #4: base case, return 1
   factorial(2) call #3: returning 2
  factorial(3) call #2: returning 6
 factorial(4) call #1: returning 24
factorial(5) call #0: returning 120
```

New function calls are pushed on the stack.
Upon return, a function call is popped off the stack.

computing factorials iteratively in an exception handler

```
def factexcept(n):
    """
    when the recursion depth is exceeded
    the factorial of n is computed iteratively
    """
    if n <= 1:
        return 1
    else:
        try:
            return n*factexcept(n-1)
        except RuntimeError:
            f = 1
            for i in range(2, n+1):
                f = f*i
            return f
```

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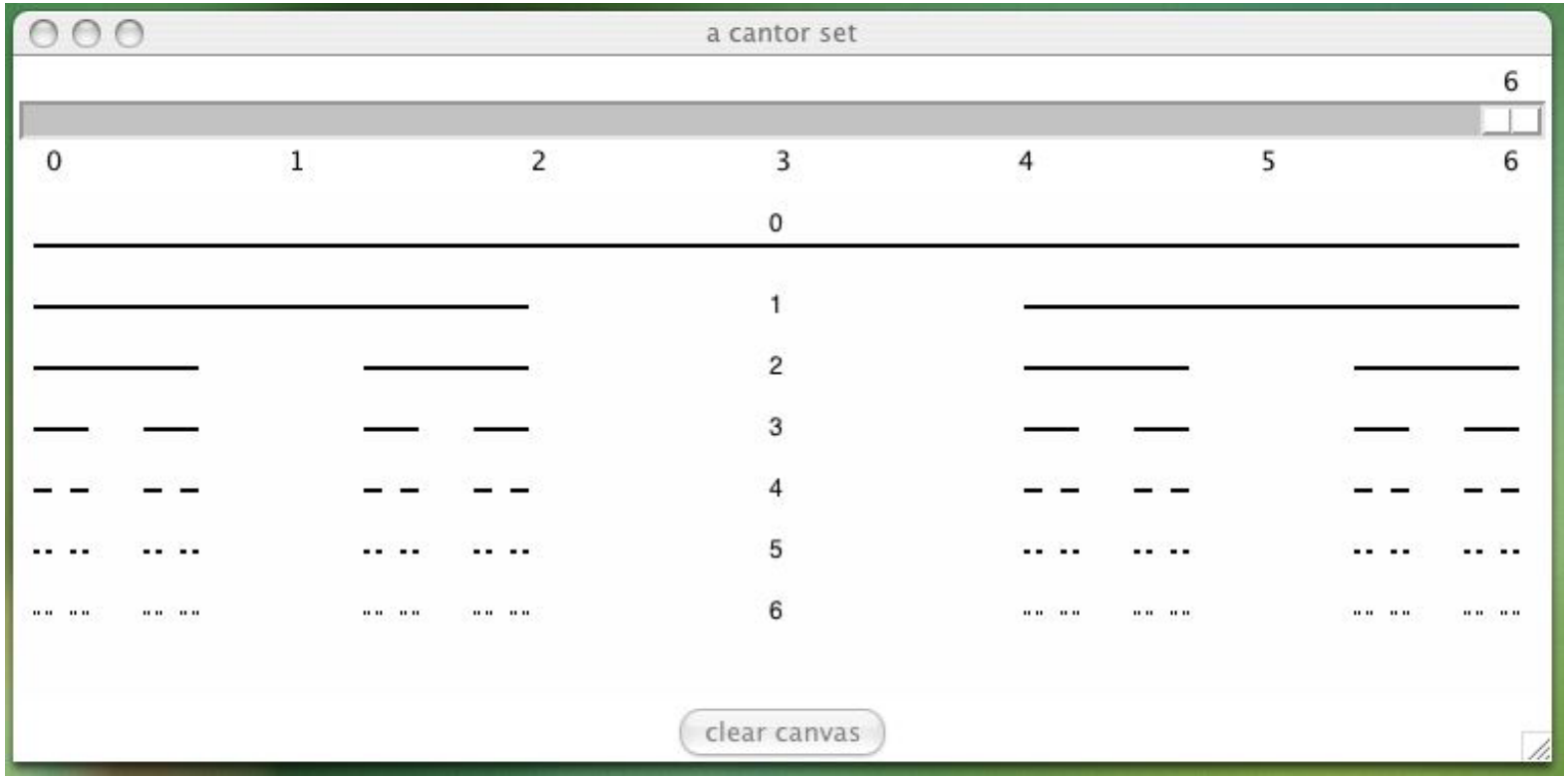
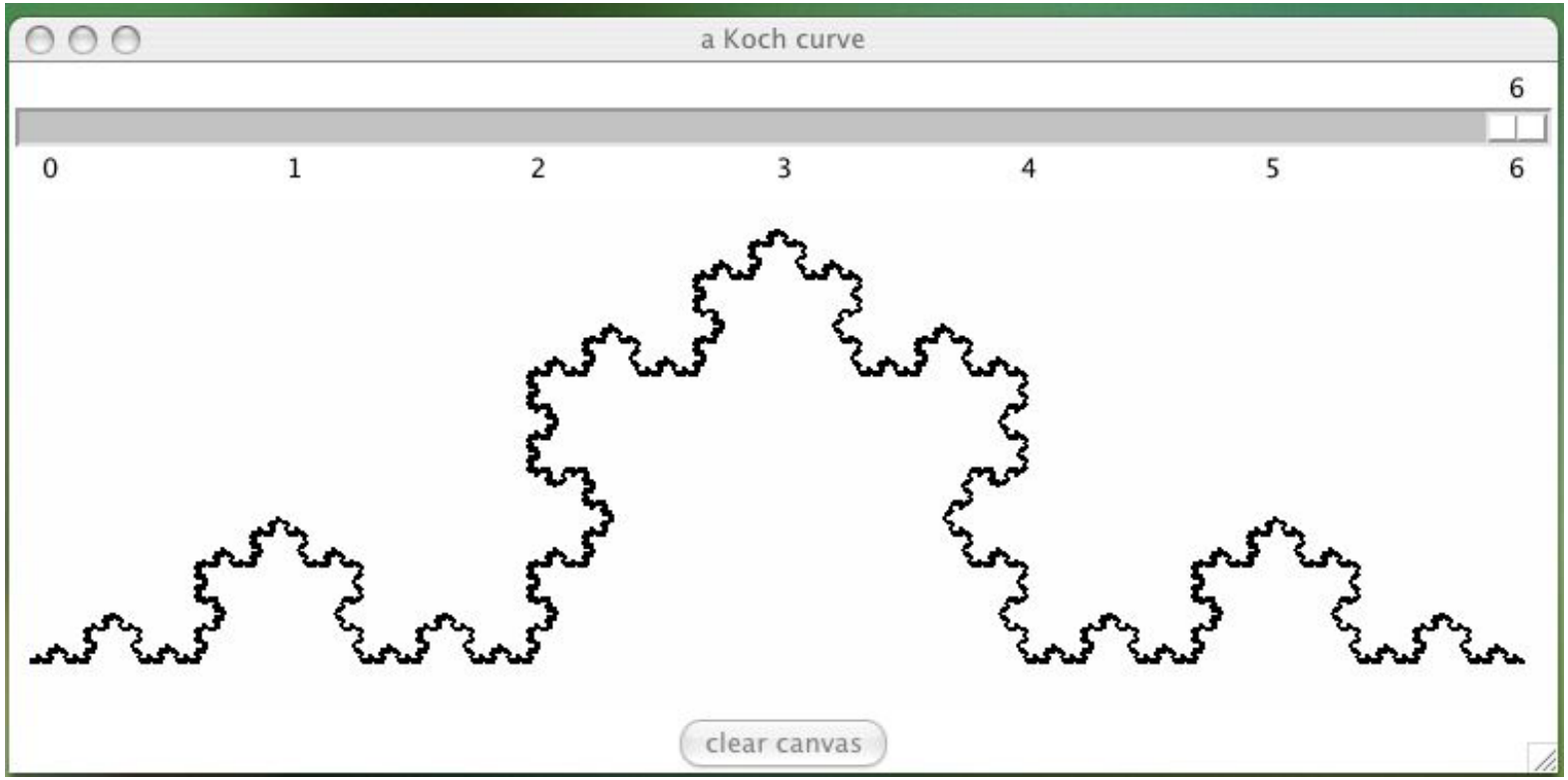
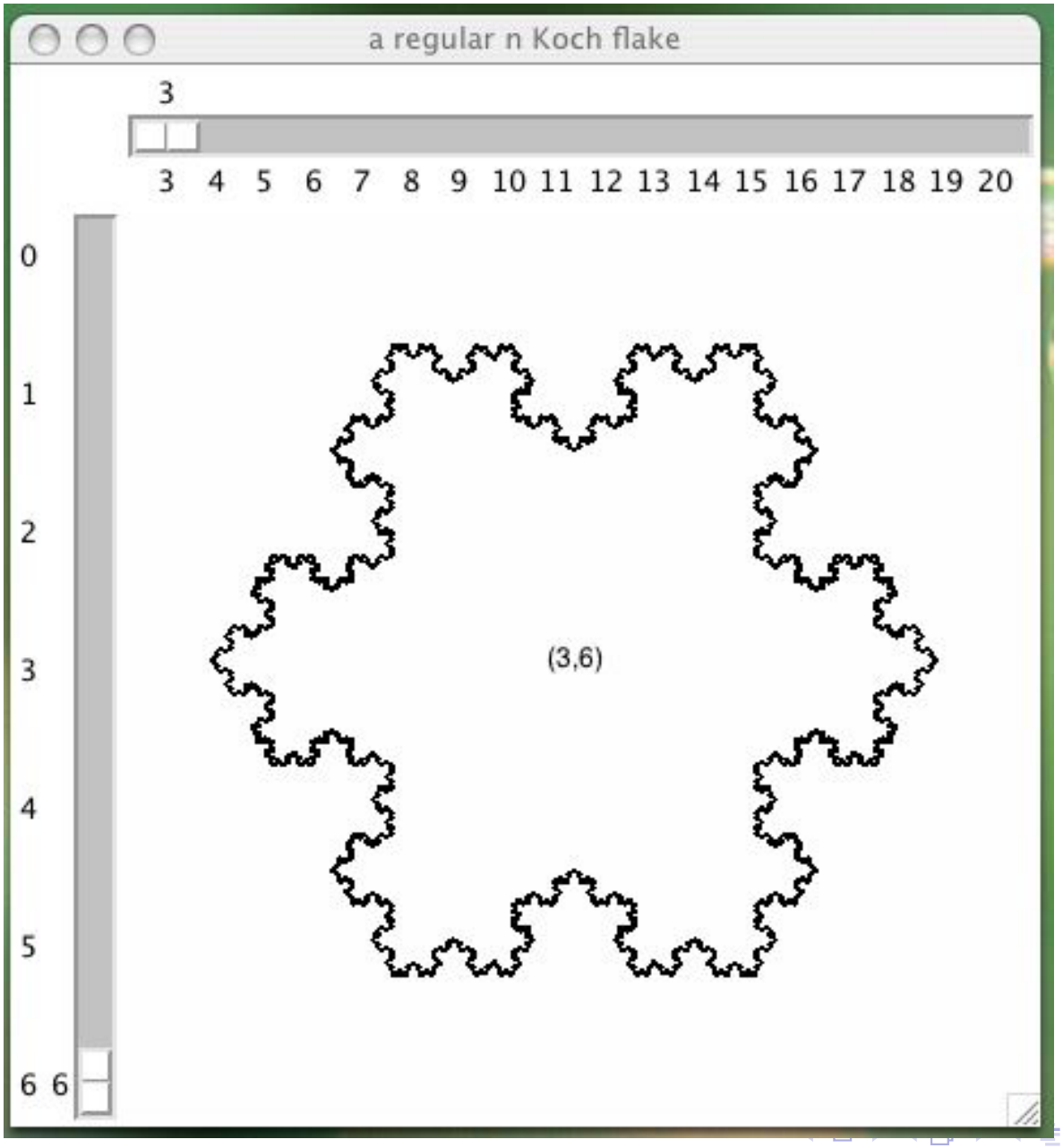
A Cantor Set



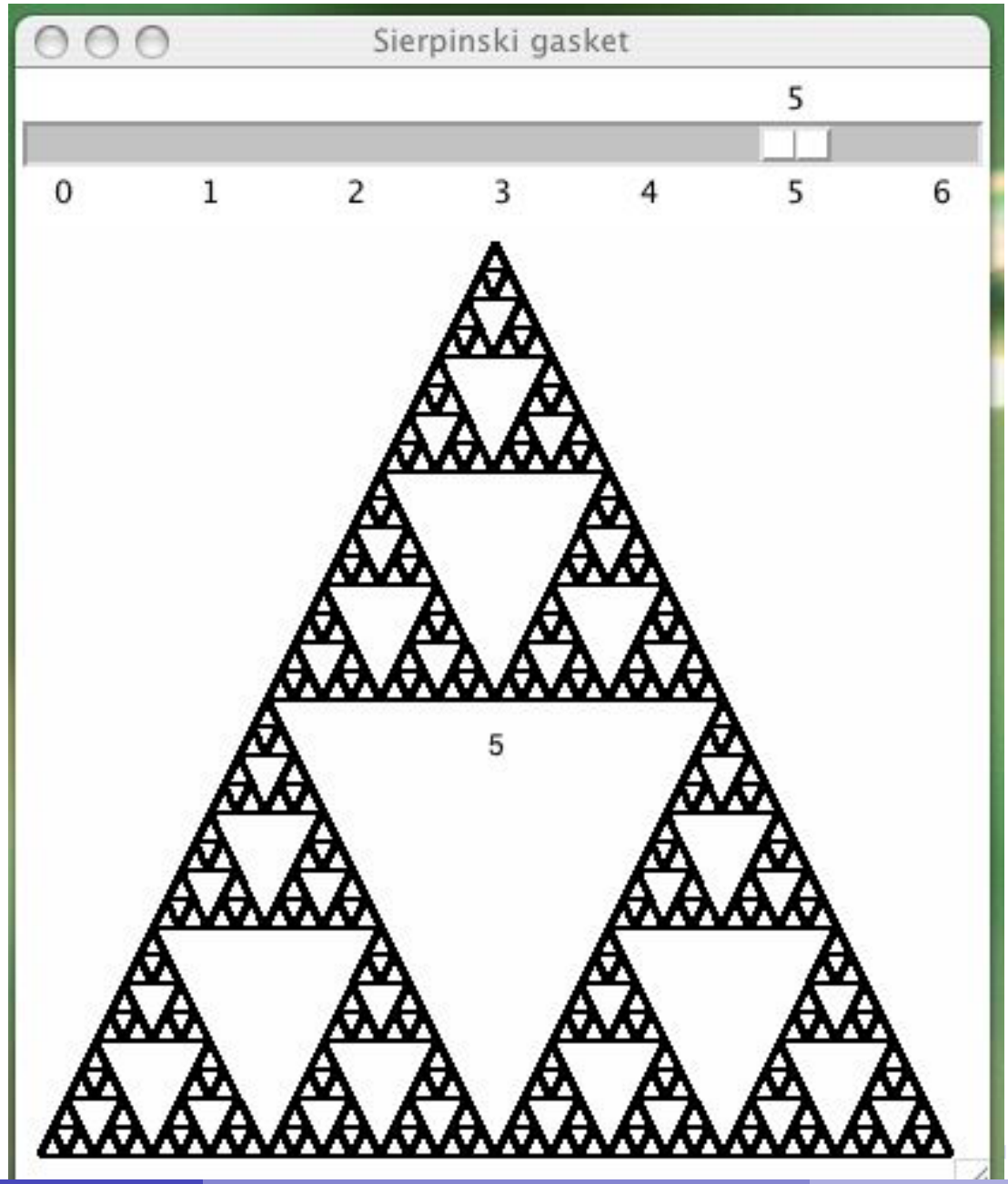
A Koch Curve



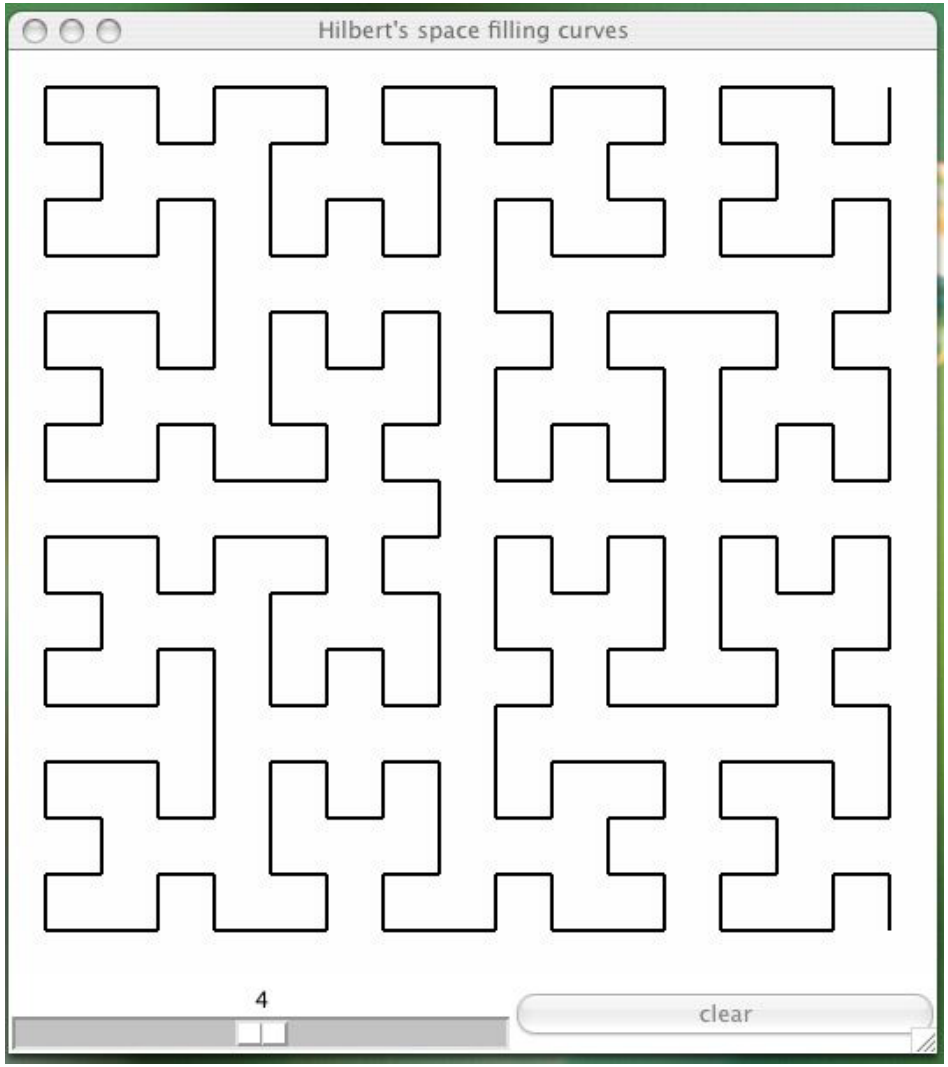
A Koch Flake


a regular n Koch flake
3
3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
0
1
2
3
4
5
6 6
(3,6)

A Sierpinski Gasket



A GUI for Hilbert Curves



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The Cantor Set

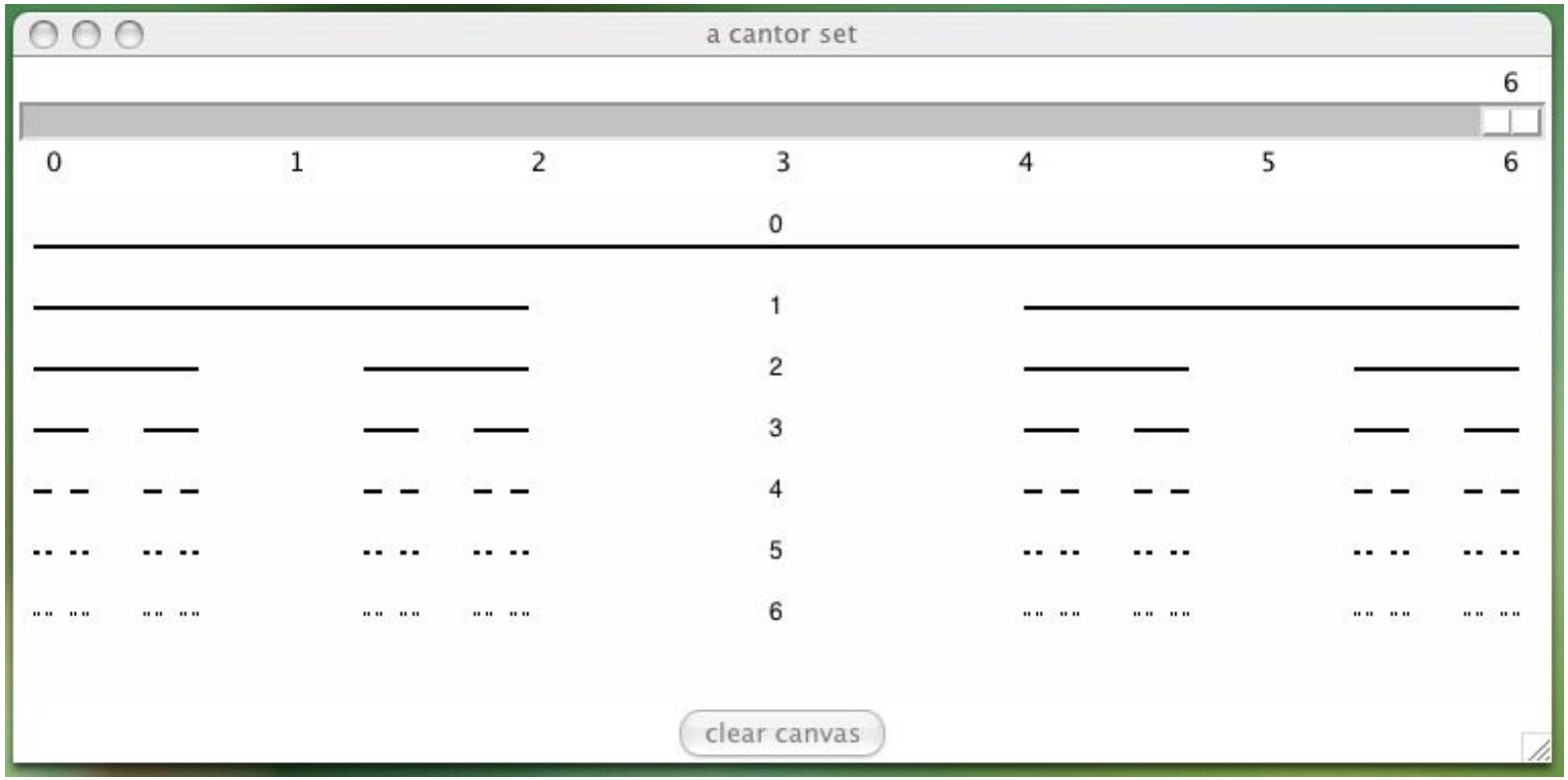
The Cantor set is defined by three rules:

1. take the interval $[0, 1]$;
2. remove the middle part third of the interval;
3. repeat rule 2 on the first and third part.

The Cantor set is infinite, to visualize at level n:

- $n = 0$: start at $[0, 1]$;
- $n > 0$: apply rule 2 n times.

GUI for a Cantor Set



the class `CantorSet`

```
from Tkinter import Tk, Canvas, ALL
from Tkinter import Button, IntVar, Scale

class CantorSet(object):
    """
    GUI to draw a Cantor set on canvas.
    """
    def __init__(self, wdw, N):
        """
        A Cantor set with N levels.
        """
    def draw_set(self, val):
        """
        Draws a Cantor set.
        """
    def clear_canvas(self):
        """
        Clears the entire canvas.
        """
```

the layout of the GUI

```
def __init__(self, wdw, N):
    """
    A Cantor set with N levels.
    """
    wdw.title('a cantor set')
    self.dim = 3**N+20
    self.n = IntVar()
    self.scl = Scale(wdw, orient='horizontal', \
        from_=0, to=N, tickinterval=1, \
        length=self.dim, variable=self.n, \
        command=self.draw_set)
    self.scl.grid(row=0, column=0)
    self.scl.set(0)
    self.cnv = Canvas(wdw, width=self.dim, \
        height=self.dim/3, bg='white')
    self.cnv.grid(row=1, column=0)
    self.btt = Button(wdw, text="clear canvas", \
        command=self.clear_canvas)
    self.btt.grid(row=2, column=0)
```

methods `clear_canvas` and `draw_set`

The method `clear_canvas()` is triggered by a Button.

```
def clear_canvas(self):
    """
    Clears the entire canvas.
    """
    self.cnv.delete(ALL)
```

The method `DrawSet()` is triggered by a Scale.

```
def draw_set(self, val):
    """
    Draws a Cantor set.
    """
    nbr = int(val)
    self.cantor(10, self.dim-10, 30, val, nbr)
```

The method `cantor` is recursive.

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parameters of a recursive function

```
def cantor(self, lft, rgt, hgt, txt, lvl):
    """
    Draws a line from lft to rgt, at height hgt
    txt is a string, int(txt) equals the number
    of times the middle third must be removed
    lvl is level of recursion, start at lvl = int(txt)
    """
```

The parameters `lft`, `rgt`, and `hgt` define
the line segment from (`lft`,`hgt`) to (`rgt`,`hgt`).

The parameter `txt` is the value passed via the Scale,
as text string, `txt` is also put on Canvas.

Initially: `lvl = int(txt)`.
With every recursive call, `lvl` is decremented by 1.

a recursive drawing algorithm

The `lvl` in `cantor(self, lft, rgh, hgt, txt, lvl)` controls the recursion.

At `lvl = 0`, the line segment from (`lft`, `hgt`) to (`rgh`, `hgt`) is drawn.

For `lvl > 0`, we compute left and right limit of the middle third of [`lft`, `rgh`], respectively denoted by `nlf` and `nrg` as

$$\texttt{nlf} = \texttt{lft} + (\texttt{rgh} - \texttt{lft})//3 = (2\texttt{lft} + \texttt{rgh})//3$$
$$\texttt{nrg} = \texttt{rgh} - (\texttt{rgh} - \texttt{lft})//3 = (\texttt{lft} + 2\texttt{rgh})//3$$

Then we make two recursive calls:

```
self.cantor(lft, nlf, hgt+30, txt, lvl-1)
self.cantor(nrg, rgh, hgt+30, txt, lvl-1)
```

code for `cantor`

```
def cantor(self, lft, rgt, hgt, txt, lvl):
    " ... "
    if lvl == 0:                # draw line segment
        self.cnv.create_line(lft, hgt, rgt, hgt, width=2)
    else:
        nlf = (2*lft+rgt)//3
        nrg = (lft+2*rgt)//3
        self.cantor(lft, nlf, hgt+30, txt, lvl-1)
        self.cantor(nrg, rgt, hgt+30, txt, lvl-1)
    if lvl == int(txt):         # put text string
        xctr = self.dim//2
        if txt == '0':
            self.cnv.create_text(xctr, hgt-10, text=txt)
        else:
            self.cnv.create_text(xctr, hgt+lvl*30, \
                text=txt)
```

some closing observations

What comes next?

- mcs 275: programming tools and file management
- mcs 320: symbolic computation
- mcs 360: data structures