Recursion versus Iteration

The Towers of Hanoi

recursive problem solving
a recursive Python function
tracing: exponential time

The Fibonacci Numbers

a simple recursion
an iterative algorithm

Exponential Complexity and Cost

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MCS 275 Lecture 8
Programming Tools and File Management
Jan Verschelde, 1 February 2008

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an ancient mathematical puzzle

Input: disks on a pile, all of varying size, no larger disk sits above a smaller disk, and two other empty piles.

Task: move the disks from the first pile to the second, obeying the following rules:

1. move one disk at a time,
2. never place a larger disk on a smaller one,

you may use the third pile as buffer.

The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
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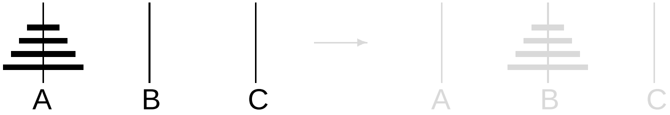

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The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

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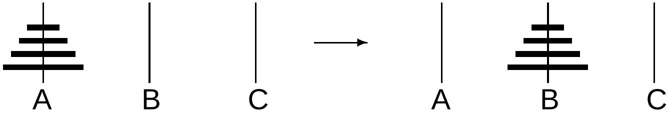

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recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

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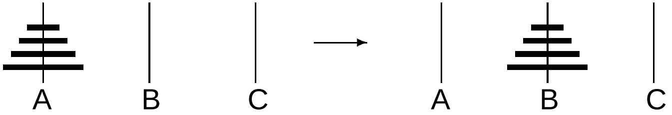

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Assume we know how to move a stack with one disk less.

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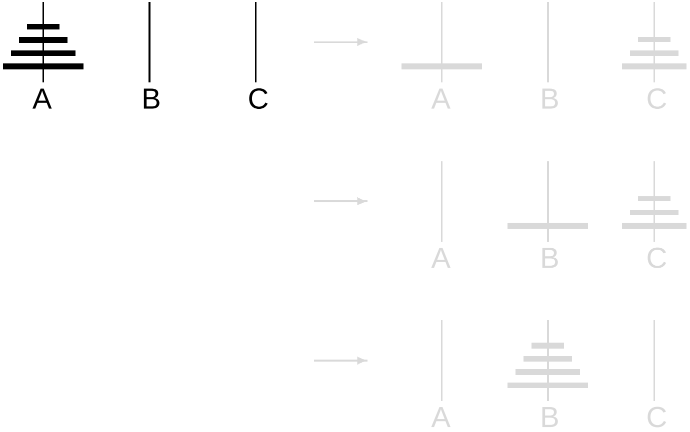

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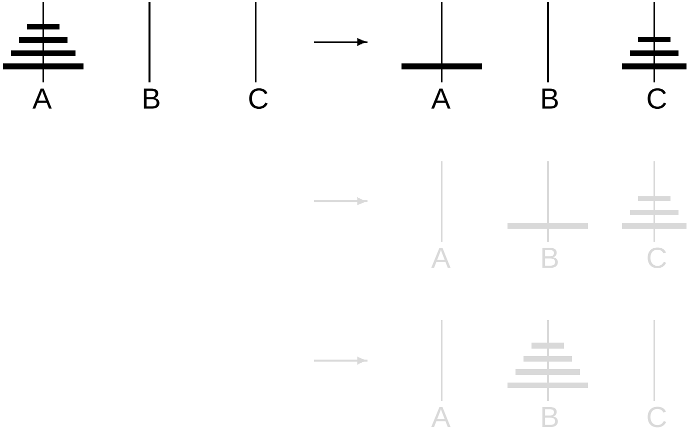

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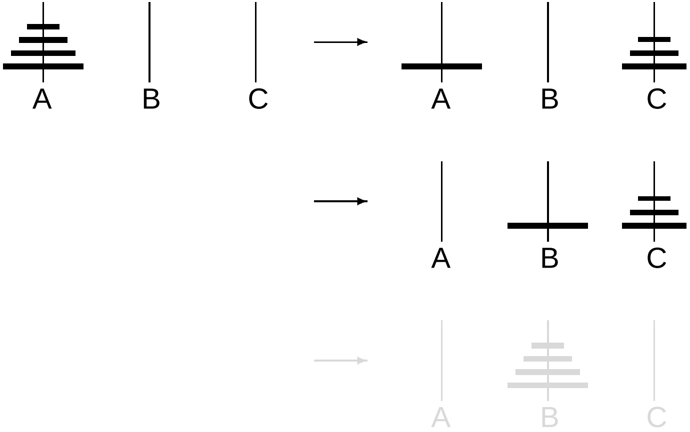

The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

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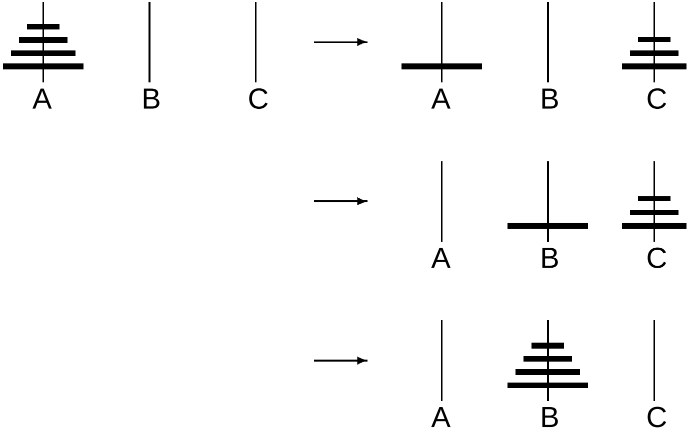

The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

A recursive Algorithm

Base case: move one disk from A to B.

To move n disks from A to B:

Move $n-1$ disks from A to C using B as auxiliary pile

Move n-th disk from A to B

Move $n-1$ disks from C to B using A as auxiliary pile

The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

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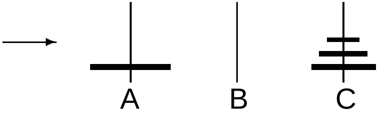

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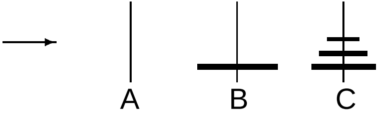

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The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

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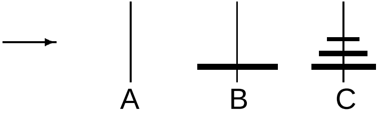

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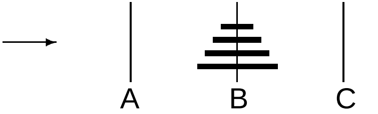

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The Towers of Hanoi

- recursive problem solving
- a recursive Python function
- tracing: exponential time

The Fibonacci Numbers

- a simple recursion
- an iterative algorithm

Exponential Complexity and Cost

- recursion versus iteration

The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

Lists as Stacks

In a stack, we remove only the top element (*pop*), and add only at the top (*push*).

A pile of 4 disks of decreasing size:

```
>>> A = range(1,5)
>>> A
[1, 2, 3, 4]
```

To remove the top element:

```
>>> A.pop(0)
1
>>> A
[2, 3, 4]
```

To put an element on top:

```
>>> A.insert(0,1)
[1, 2, 3, 4]
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The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time

The Fibonacci Numbers
a simple recursion
an iterative algorithm

Exponential Complexity and Cost
recursion versus iteration

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The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

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The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

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A recursive Python Function

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def Hanoi(n,A,B,C):
    """
    moves n disks from A to B, C is auxiliary
    returns the tuple (A,B,C)
    """
    if n == 1:
       # move disk from A to B
       B.insert(0,A.pop(0))
    else:
       # move n-1 disks from A to C, B is auxiliary
       (A,C,B) = Hanoi(n-1,A,C,B)
       # move n-th disk from A to B
       B.insert(0,A.pop(0))
       # move n-1 disks from C to B, A is auxiliary
       (C,B,A) = Hanoi(n-1,C,B,A)
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The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

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The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

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The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

The Towers of Hanoi
recursive problem solving
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an iterative algorithm
Exponential Complexity and Cost
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The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

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The Towers of Hanoi
- recursive problem solving
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The Fibonacci Numbers
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Exponential Complexity and Cost
- recursion versus iteration

The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

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The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

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The Towers of Hanoi
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Exponential Complexity and Cost
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The Towers of Hanoi
recursive problem solving
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The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

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an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

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The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

Tracing the Execution

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The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

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The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

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The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

Tracing the Execution

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The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

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Extra Code for Tracing

recursion level, count #moves

```
def Hanoi(n,A,B,C,k,m):
    """
    moves n disks from A to B, C is auxiliary
    k is recursion level, m counts # moves
    writes status of piles after each move
    returns the tuple (A,B,C,m)
    """
```

in `main()`:

```
    n = input('Give number of disks : ')
    A = ('A',range(1,n+1))
    B = ('B',[])
    C = ('C',[])
    (A,B,C,m) = Hanoi(n,A,B,C,0,0)
```

As the roles of the piles shift, we need to maintain their names when printing their contents.

The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

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The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

Extended Function Hanoi

```
def Hanoi(n,A,B,C,k,m):
   "..."
   if n == 1:
      # move disk from A to B
      m = m + 1
      B[1].insert(0,A[1].pop(0))
      write(k,m,n,A,B,C)
   else:
      # move n-1 disks from A to C, B is auxiliary
      (A,C,B,m) = Hanoi(n-1,A,C,B,k+1,m)
      # move n-th disk from A to B
      m = m + 1
      B[1].insert(0,A[1].pop(0))
      write(k,m,n,A,B,C)
      # move n-1 disks from C to B, A is auxiliary
      (C,B,A,m) = Hanoi(n-1,C,B,A,k+1,m)
   return (A,B,C,m)
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The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

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The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

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      (C,B,A,m) = Hanoi(n-1,C,B,A,k+1,m)
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```

Writing the States

Pile A is a tuple `(A[0],A[1])`:
`A[0]` is name, `A[1]` is list.

```
def write_piles(s,A,B,C):
   "writes contents of piles, after s"
   sA = '%s = %s' % (A[0],A[1])
   sB = '%s = %s' % (B[0],B[1])
   sC = '%s = %s' % (C[0],C[1])
   print s, sA, sB, sC

def write(k,m,n,A,B,C):
   "writes contents of piles"
   s = k*' '
   s = s + 'move %d, n = %d :' % (m,n)
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The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

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Exponential Execution Time

Observe: to move n disks, we need

$n = 1 \to 1$ move $\quad n = 2 \to 3$ moves

$n = 3 \to 7$ moves $\quad n = 4 \to 15$ moves ...

Let $T(n)$ count number of moves for n disks:

$$T(1) = 1 \qquad T(n) = 2T(n-1) + 1.$$

Solving the recurrence relation:

$$\begin{aligned} T(n) &= 2T(n-1) + 1 \\ &= 2(2T(n-2) + 1) + 1 \\ &= 2^k T(n-k) + 2^{k-1} + \cdots + 2 + 1 \\ &= 2^{n-1} + 2^{n-2} + \cdots + 2 + 1 \\ &= 2^n - 1 \end{aligned}$$

The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

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The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

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Recursion versus Iteration

The Towers of Hanoi
- recursive problem solving
- a recursive Python function
- tracing: exponential time

The Fibonacci Numbers
- a simple recursion
- an iterative algorithm

Exponential Complexity and Cost
- recursion versus iteration

The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

The Fibonacci Numbers

The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

The n-th Fibonacci number F_n is defined as

$$F_0 = 0, F_1 = 1, \quad n > 1 : F_n = F_{n-1} + F_{n-2}.$$

```
def Fibonacci(n):
   """
   returns n-th n-th Fibonacci number
   """
   if n == 0:
      return 0
   elif n == 1:
      return 1
   else:
      return Fibonacci(n-1) + Fibonacci(n-2)
```

The Fibonacci Numbers

The n-th Fibonacci number F_n is defined as

$$F_0 = 0, F_1 = 1, \quad n > 1 : F_n = F_{n-1} + F_{n-2}.$$

```
def Fibonacci(n):
   """
   returns n-th n-th Fibonacci number
   """
   if n == 0:
      return 0
   elif n == 1:
      return 1
   else:
      return Fibonacci(n-1) + Fibonacci(n-2)
```


The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

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The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

Computing Fibonacci(5)

```
$ python fibonacci.py
Give n : 5
F(5) = F(4) + F(3)
 F(4) = F(3) + F(2)
  F(3) = F(2) + F(1)
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   F(1) = 1
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F(5) = 5
number of calls : 25
```

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The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

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The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

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The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

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F(5) = 5
number of calls : 25
```

Computing Fibonacci(5)

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The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

Computing `Fibonacci(5)`

```
$ python fibonacci.py
Give n : 5
F(5) = F(4) + F(3)
 F(4) = F(3) + F(2)
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   F(2) = F(1) + F(0)
    F(1) = 1
    F(0) = 0
   F(1) = 1
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   F(1) = 1
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 F(3) = F(2) + F(1)
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```

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The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

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The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

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The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

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The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

Tracing the Execution

```
def Fibotrace(n,k,c):
    """
    returns (f,c) f is the n-th Fibonacci number
    and c counts the number of function calls
    prints execution trace using control parameter k
    """
    s = k*' ' + 'F(%d) = ' % n
    if n == 0:
        print s + '0'
        return (0,c)
    elif n == 1:
        print s + '1'
        return (1,c)
    else:
        print s + 'F(%d) + F(%d)' % (n-1,n-2)
        (f1,c1) = Fibotrace(n-1,k+1,c+1)
        (f2,c2) = Fibotrace(n-2,k+1,c+1)
        return (f1+f2,c1+c2)
```


The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

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The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

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Recursion versus Iteration

The Towers of Hanoi
- recursive problem solving
- a recursive Python function
- tracing: exponential time

The Fibonacci Numbers
- a simple recursion
- an iterative algorithm

Exponential Complexity and Cost
- recursion versus iteration

Fibonacci with an iterative Algorithm

```
def F(n):
    """
    iterative way for n-th Fibonacci number
    """
    if n == 0:
        return 0
    else:
        a = 0
        b = 1
        for k in range(2,n+1):
            c = a + b
            a = b
            b = c
        return b
```

Fibonacci with an iterative Algorithm

```
def F(n):
    """
    iterative way for n-th Fibonacci number
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The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

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The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time

The Fibonacci Numbers
a simple recursion
an iterative algorithm

Exponential Complexity and Cost
recursion versus iteration

Exponential Complexity and Cost

The towers of Hanoi problem is hard *no matter what* algorithm is used. Its ***complexity*** is exponential.

The first recursive computation of the Fibonacci numbers took longs, its ***cost*** is exponential.

If the number of function calls exceeds the size of the results, we better use an iterative formulation.

Using a stack to store the function calls, every recursive program can be transformed into an iterative one.

Background material for this lecture:

- Chapter 8 of *The Art & Craft of Computing*, in particular: read §8.3 for execution details.

The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

Exponential Complexity and Cost

The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
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Exponential Complexity and Cost
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The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

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recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

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recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

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recursive problem solving
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The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

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The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration

Exponential Complexity and Cost

The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
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Exercises

1. Write a Python function `F(n)` which returns a list of the first n Fibonacci numbers.
2. We define the Harmonic numbers H_n as $H_1 = 1$ and $H_n = H_{n-1} + 1/n$. Write a recursive function for H_n.
3. Extend the recursive function for H_n (see above) with a parameter to keep track of the number of function calls. Write an iterative function for H_n.
4. Write an iterative version for the function `is_palindrome()` of Lecture 6.
5. Design a GUI to show the moves to solve the problem of the towers of Hanoi.

The Towers of Hanoi
recursive problem solving
a recursive Python function
tracing: exponential time
The Fibonacci Numbers
a simple recursion
an iterative algorithm
Exponential Complexity and Cost
recursion versus iteration