

## Maple Lecture 11. Rational Expressions and Conversions

In this lecture we encounter for the first time the normalization problem and relate this to the problem of “expression swell”. We also investigate the best way to evaluate expressions. For dense polynomials we can do not much better than converting to the Horner form, while for rational expressions, we save operations when we convert to continued fractions.

The material in this lecture corresponds to [1, Sections 5.3 and 5.4].

### 11.1 Rational Expressions

Besides polynomials, Maple recognizes the type `ratpoly`:

```
[> p := x^3-1; q := x^2 - 1;
[> f := p/q;
[> whattype(p); type(f, ratpoly);
```

Note that Maple does not simplify automatically. We have to normalize with `normal`:

```
[> normal(f);
```

The normalization of rational expressions clears all common factors from the numerator and the denominator. There are many other representations of the same expression. For instance, we can factor the numerator:

```
[> f = factor(numer(f))/denom(f);
```

or the denominator :

```
[> f = numer(f)/factor(denom(f));
```

or factor both :

```
[> f = factor(numer(f))/factor(denom(f));
```

Here the simplification happened automatically. Let us try to prevent the normalization:

```
[> nf := factor(numer(f)); df := factor(denom(f));
```

To prevent the cancellation of the common factor we can freeze the numerator:

```
[> ff := freeze(nf)/df;
[> thaw(numer(ff));
```

If you ask for a good reason why Maple does not normalize automatically, then here is an example:

```
[> fd := (x^1000-1)/(x-1);
```

Can you explain now why Maple does not normalize automatically?

Finding a normal form for a symbolic expression is very important to decide whether two expressions are the same, but sometimes a normal form may be computationally too expensive.

## 11.2 Conversions

For efficient evaluation of polynomials or rational expressions, we may wish to convert to the Horner form or to continued fractions respectively.

```
[> p := sum('a[i]*x^i', 'i'=0..5);
[> with(codegen):                # we need the expression cost
[> cost(p);                       # cost for evaluation
[> hp := convert(p, 'horner');
[> cost(hp);
```

Observe though that the C code generation does not follow this Horner rule:

```
[> C(p, 'optimized');
```

For rational expressions, we convert to continued fractions:

```
[> q := sum('b[i]*x^i', 'i'=0..5);
[> pq := p/q;
[> cost(pq);
[> #cpq := convert(pq, 'confrac', x); # this did not return...
```

Let us try it with random coefficients :

```
[> for i from 0 to 5 do
[>   a[i] := rand(); b[i] := rand();
[> end do;
[> cpq := convert(pq, 'confrac', x);
[> cost(cpq);
```

With continued fractions, we have replaced multiplications by divisions. Since there are fewer divisions than the original number of multiplications, one would prefer the continued fraction form. However, observe the growth of the numbers...

Finally we compare with cost of using the Horner form on numerator and denominator:

```
[> hpq := convert(numer(pq), 'horner')/convert(denom(pq), 'horner');
[> cost(hpq);
```

We close with another type of conversion. To integrate rational expressions, Maple converts to partial fractions:

```
[> convert(pq, 'parfrac', x, complex);
```

Observe that the choice of the number field (in this case complex) drastically changes the outcome.

### 11.3 Assignments

1. Generate a random dense univariate polynomial of degree 5 (use `p := randpoly(x, degree=5, dense);`). Convert this polynomial into Horner form and compare the cost of evaluating `p` in its original form to the cost of evaluating its Horner form. Does the order of terms in the polynomial `p` matter? Illustrate by sorting `p` in ascending and descending order of degrees and computing the cost in each case.

Write a formula for the cost for polynomials of general degree  $d$ , i.e.: how many additions (or subtractions) and multiplications does it take to evaluate respectively a polynomial of degree  $d$  and its corresponding Horner form?

2. Assign to  $r$  the rational expression  $\frac{4x^2+x}{x^2+4}$  and use `op` to draw the expression tree of  $r$ .

3. Consider the rational expression  $\frac{x^4+x^3-4x^2-4x}{x^4+x^3-x^2-x}$ .

Transform the expression into:

- (a)  $\frac{(x+2)(x+1)(x-2)}{x^3+x^2-x-1}$
- (b)  $\frac{x^4+x^3-4x^2-4x}{x(x-1)(x+1)^2}$
- (c)  $\frac{(x+2)(x-2)}{(x-1)(x+1)}$
- (d)  $\frac{x^2}{(x-1)(x+1)} - 4\frac{1}{(x-1)(x+1)}$

4. Consider the same rational expression as in the previous exercise.

- (a) Write it as a continued fraction.
- (b) Compute a partial fraction decomposition.

5. Consider  $r = \frac{6z^8+5z^4+1}{3z^8+4z^4+4}$ .

- (a) How many operations does it take to evaluate  $r$  in the form above?  
Give the Maple command(s) used to obtain this answer.
- (b) Convert  $r$  into a continued fraction form and calculate again the number of operations it takes to evaluate  $r$  in this form. List all Maple commands used.
- (c) What is the most efficient way to evaluate  $r$ ?  
How many operations do you need in this case?

6. Give the Maple command(s) to convert  $\frac{1}{2x^3+x^2+2x}$  into  $\frac{1}{2x} + \frac{-\frac{1}{4} + \frac{1}{60}I\sqrt{15}}{x + \frac{1}{4} - \frac{1}{4}I\sqrt{15}} + \frac{-\frac{1}{4} - \frac{1}{60}I\sqrt{15}}{x + \frac{1}{4} + \frac{1}{4}I\sqrt{15}}$ .

### References

- [1] A. Heck. *Introduction to Maple*. Springer-Verlag, third edition, 2003.