

## Maple Lecture 27. Differential Equations

In a crude overview, we can say that Maple offers three different types of solutions to ODEs:

1. analytic solutions, in the form of an explicit formula for the solution;
2. the solution expressed as a series; and
3. a numerical solution to the ODE.

While an analytic solution may seem the most desirable solution, we must know that (similar to integration where formal antiderivatives do not always exist) a nice formula for the solution may not exist.

The command **dsolve** is one of the most useful solvers in Maple as differential equations have lots of applications, see [1, 2]. This lecture is a selection of the material in [3, Chapter 17].

### 27.1 Analytic Solutions

If we have no preference for the type of method, we just call **dsolve** without any special parameters. We define  $y$  as a function of  $x$  by the following differential equation:

```
[> ODE := x*diff(y(x),x) = y(x)*ln(x*y(x)) - y(x);
[> s := dsolve(ODE,y(x));
```

Maple returns a formal solution where  $\_C1$  is any arbitrary complex constant. How can we verify the solution?

```
[> yx := rhs(s);
[> syx := subs(y(x)=yx,ODE);
[> simplify(lhs(syx));
[> simplify(rhs(syx));
```

### 27.2 Taylor Series Methods

With this method, the solution is developed as a Taylor series and we use the conditions imposed by the ODE to find values for the coefficients of the Taylor series expansion.

Let us look at the pendulum, defined by a second-order differential equation :

```
[> diffeq := l*diff(theta(t),t$2) = -g*sin(theta(t));
```

The  $l$  in the equation is the length of the pendulum,  $\theta(t)$  is the angle measuring the deviation from its vertical position and  $g$  is the gravitational constant. Since this is a second-order equation, we need two initial conditions. At time  $t=0$ , the pendulum is in vertical position, so  $\theta(0) = 0$ , and its velocity is some  $v[0]$ .

```
[> inits := theta(0) = 0, D(theta)(0) = v[0];
```

The ODE is thus defined by a differential equation and two initial conditions:

```
[> ODE := {diffeq,inits};
```

Observe the very important difference in the use of **diff** and **D**. In the equation, we set up a formula and use **diff**. In the initial condition, we regard  $\theta$  as a function of time and must use **D**.

```
[> sol := dsolve(ODE,theta(t),series);
```

To crank up the order of approximation, we change the variable **Order**. See the lecture on series to turn this into a function.

### 27.3 Numerical Solutions

Using numerical algorithms, we can study more advanced oscillators than just the pendulum. Let us take the van der Pol equation:  $y'' - (1 - y^2) * y' + y = 0$ , with  $y(0) = 0, y'(0) = -0.1$ .

```
[> diffeq := diff(y(t),t$2) - (1-y(t)^2)*diff(y(t),t) + y(t) = 0;
[> inits := y(0) = 0, D(y)(0) = -0.1;
[> ODE := {diffeq,inits};
[> sol := dsolve(ODE,y(t),type=numeric);
[> sol(0.1);
```

We see that we get a list of three elements when we evaluate the solution at any given time. But if we are only interested in the evolution of  $y(t)$  in function of time, we need to select from the output of dsolve the right function for plotting purposes:

```
[> solfun := t -> rhs(sol(t)[2]);
[> solfun(0.1);
[> plot(solfun,0..20);
```

### 27.4 DEtools

The package DEtools offers a wide range of graphical tools to study ODEs:

```
[> with(DEtools);
```

Here we take the van der Pol equation again and make a “phase portrait” of two curves defined by the equation. We first transform our second-order equation into a system of two first-order linear equations:

```
[> ODEs := diff(y(t),t) = yy(t), diff(yy(t),t) - (1-y(t)^2)*yy(t) + y(t)=0;
```

If we want to see two trajectories, we cook up two sequences of initial conditions:

```
[> inits1 := y(0) = 0, yy(0) = -0.1;
[> inits2 := y(0) = 0.1, yy(0) = 0;
[> DEplot({ODEs}, [y(t),yy(t)], t=0..10, [[inits1],[inits2]]);
```

## 27.5 Assignments

1. Show that the solution  $y(x)$  to the equation

$$\sqrt{a^2 - y^2} - a \ln \left( a + \sqrt{a^2 + y^2} \right) + a \ln(y) + x = C$$

is a solution of the differential equation

$$y' = -\frac{y}{\sqrt{a^2 - y^2}}.$$

2. Solve the following ODE:

$$3y^2y' + 16x = 12xy^2$$

and verify the solution you find.

3. Solve the following initial value problem:

$$y'' - y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

4. The Chebyshev polynomials  $T_n(x)$  satisfy the following differential equation:

$$(1 - x^2)T_n''(x) - xT_n'(x) + n^2T_n(x) = 0.$$

- (a) Apply **dsolve** to the differential equation and interpret what Maple returns to you.

Note:  $T_n(x) = \cos(n \arccos(x))$ .

- (b) The command **orthopoly[T](25,x)** returns  $T_{25}(x)$ .

Verify that **orthopoly[T](25,x)** satisfies the differential equation.

5. Consider the following system of differential equations:

$$\begin{cases} \frac{d}{dt}r(t) &= & 1.1 r(t) &-& 0.5 r(t)f(t) \\ \frac{d}{dt}f(t) &= & -0.75 f(t) &+& 0.25 r(t)f(t) \end{cases} \quad r(0) = 3, f(0) = 2.$$

This system can be used to model the evolution in time  $t$  of two species, with  $f(t)$  the size of the predator population ( $f = \text{fox}$ ) and  $r(t)$  the size of the prey population ( $r = \text{rabbit}$ ).

- (a) Use **dsolve** to define two numerical procedures **rabbits** and **foxes** which are functions of  $t$ : **rabbits**( $t$ ) =  $r(t)$  and **foxes**( $t$ ) =  $f(t)$ , i.e.: **rabbits** is the solution for  $r(t)$  and **foxes** is the solution for  $f(t)$ , as defined by the given system of differential equations.

- (b) With these two functions, make a plot of the solution trajectories, for  $t$  going from 0 to 8.

## References

- [1] M.L. Abel and J.P. Braselton. *Differential Equations with Maple V*. Academic Press, second edition, 2000.
- [2] D. Barrow, A. Belmonte, A. Bogges, J. Bryant, T. Kiffe, J. Morgan, M. Rahe, S. Kirby, and M. Stecher. *Solving Differential Equations with Maple V Release 4*. Brooks/Cole Publishing Company, 1998.
- [3] A. Heck. *Introduction to Maple*. Springer-Verlag, third edition, 2003.