Evaluation and Execution

1. Evaluation
   - finite number fields
   - addition and multiplication tables
2. Execution
   - binary expression trees
   - the stack to evaluate an expression

MCS 320 Lecture 8
Introduction to Symbolic Computation
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Making a Field of Four Elements

If we work modulo 4, then $(2 \times 2) \mod 4 = 0$.

Therefore, we prefer not to work with the modulo 4 numbers.

We extend $\mathbb{Z}_2$ with a root of an irreducible polynomial:

1. Consider polynomials in x over $\mathbb{Z}_2 = \{0, 1\}$, we denote this set of polynomials by $\mathbb{Z}_2[x]$.
2. In $\mathbb{Z}_2[x]$, the polynomial $q = x^2 + x + 1$ does not factor.
3. Construct $\mathbb{Z}_2(\alpha) = \mathbb{Z}_2[x]/(x^2 + x + 1)$ as

$$\mathbb{Z}_2(\alpha) = \{ a + b\alpha \mid a, b \in \mathbb{Z}_2 \}, \quad \alpha^2 = \alpha + 1.$$

The four elements in $\mathbb{Z}_2(\alpha)$ are 0, 1, α, and $\alpha + 1$.

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Addition and Multiplication Tables

In a finite set of numbers,
we can tabulate the outcomes of all additions and multiplications.

The addition table of $\mathbb{Z}_2(\alpha) = \mathbb{Z}_2[x]/(x^2 + x + 1)$:

0	1	α	$\alpha + 1$
1	0	$\alpha + 1$	α
α	$\alpha + 1$	0	1
$\alpha + 1$	α	1	0

The multiplication table of $\mathbb{Z}_2(\alpha) = \mathbb{Z}_2[x]/(x^2 + x + 1)$:

0	0	0	0
0	1	α	$\alpha + 1$
0	α	$\alpha + 1$	1
0	$\alpha + 1$	1	α

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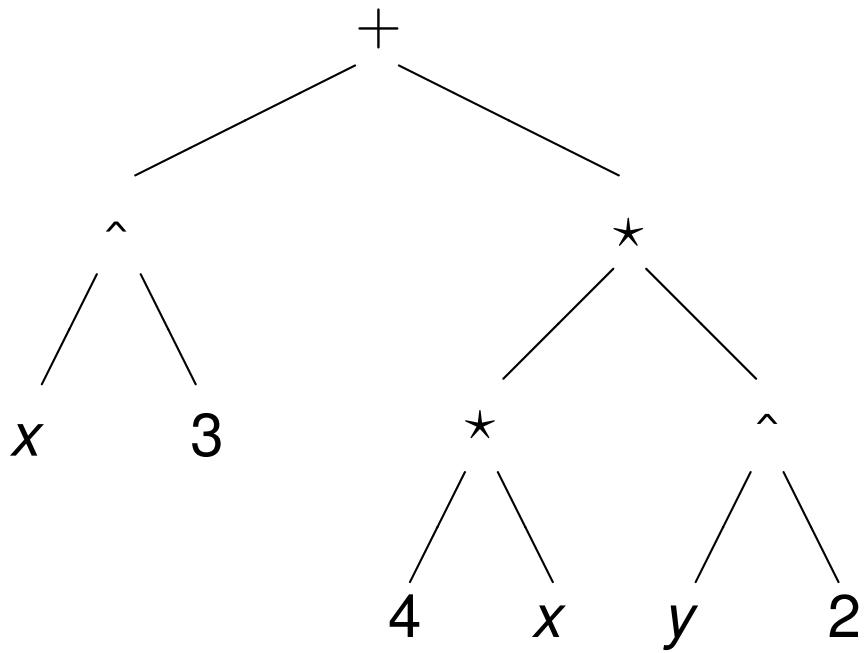
Binary Expression Trees

For fast evaluation, consider a binary expression tree:

1. The leaves of the tree are either constants or the operands.
2. The nodes of the tree are the arithmetical operators.

For example,
consider the binary expression tree for $x^3 + 4xy^2$ below:



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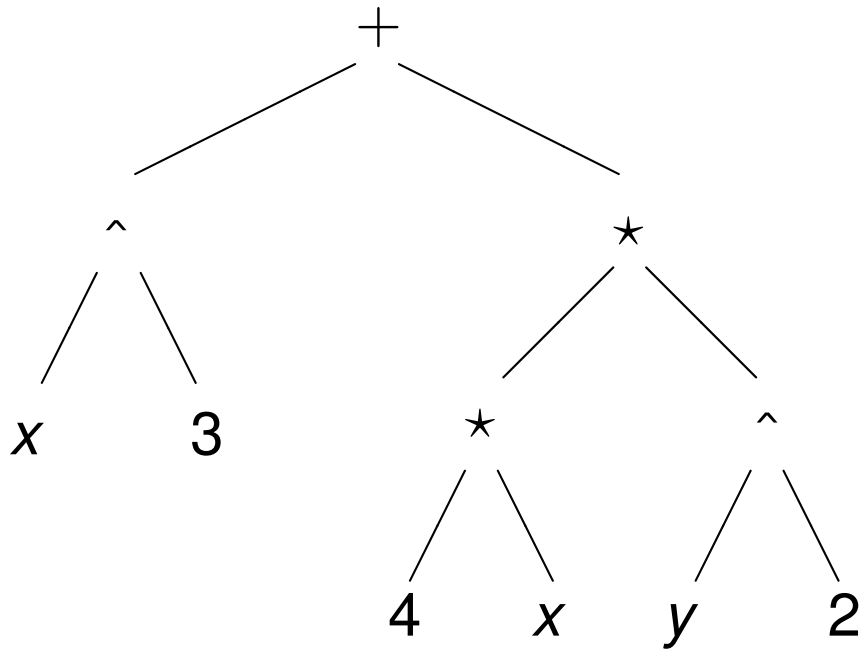
Infix, Prefix, and Postfix Tree Traversals

The expression tree for $x^3 + 4xy^2$ is drawn below.



Tree traversals lead to infix, prefix, and postfix notations.

- infix: $(x\hat{}3) + ((4 \star x) \star (y\hat{}3))$
- prefix: $+(\hat{}(x,3), \star(\star(4,x), \hat{}(y,3))$
- postfix: $x, 3, \hat{}, 4, x, \star, y, 3, \hat{}, \star, +$