Exact and Floating-Point Numbers

1. Exact and Floating-Point Numbers
   - delayed evaluation

2. Rational Approximations
   - approximations with a prescribed accuracy
   - continued fractions

MCS 320 Lecture 4
Introduction to Symbolic Computation
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Exact and Floating-Point Numbers

In symbolic computation we compute with symbols and expressions.

1. This implies that numerical approximations are delayed.
2. Rational approximations allow for exact computations.

In the context of working with approximations for π,
consider the difference between symbolic and numerical computing.

1. Computing numerically, work with an expansion, e.g.: 3.14.
2. Computing symbolically, use a rational approximation, e.g.: $22/7$.

In numerical analysis, we worry about error propagation,
in symbolic computation, expression swell is a problem.

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Approximations with a Prescribed Accuracy

To compute a rational approximation with a prescribed accuracy, execute the following two steps:

1. Evaluate a symbol in a number system.
2. Construct a rational approximation for the evaluation.

Example:

1. Evaluate π with five decimal places: 3.1416.
2. The rational approximation is then 355/113.

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Continued Fractions

Definition (continued fraction)

A *continued fraction* is

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}.$$

defined by a list of natural numbers $[a_0; a_1, a_2, \ldots, a_n]$.
The rational numbers defined by the sublists are called *the convergents* of the continued fraction.

Example: The continued fraction of $\sqrt{2}$ is defined by $[1; 2, 2, 2, 2, \ldots]$.

$$[1; 2, 2, 2, 2] \quad \text{defines} \quad 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \frac{1}{2}}}} = \frac{41}{29}$$

Consecutive Rational Approximations

The convergents of $[1; 2, 2, 2, 2]$ are $[1, 3/2, 7/5, 17/12, 41/29]$.

The first convergents for π give the consecutive rational approximations

$$3, \ \frac{22}{7}, \ \frac{333}{106}, \ \frac{355}{113}, \ \frac{103993}{33102}, \ \frac{104348}{33215}, \ \frac{208341}{66317}, \ \frac{312689}{99532}.$$