Integration and Summation

1. Integration
   - symbolic and numeric integration
   - symbolic bounds and singularities
2. Summation
   - explicit formulas for sums
   - limits of sums

MCS 320 Lecture 23
Introduction to Symbolic Computation
Jan Verschelde, 8 July 2024

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Symbolic and Numeric Integration

Consider an expression $f(x)$ in a variable x.
If a symbolic antiderivative for $f(x)$ exists, then

$$\frac{d}{dx} \int f(x)dx = f(x) = \int \frac{d}{dx} f(x)dx.$$

In many cases, there is no symbolic antiderivative
and we have to numerically approximate the definite integral

$$\int_a^b f(x)dx.$$

Theorem (Fundamental Theorem of Calculus)

Let $F(x)$ be the antiderivative of $f(x)$. Then,

$$\int_a^b f(x)dx = F(b) - F(a).$$

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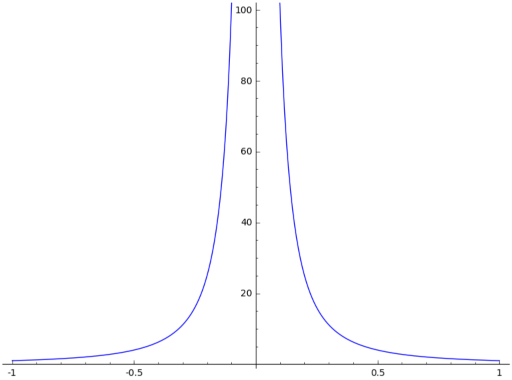
Symbolic Bounds and Singularities

$$\int_a^b \frac{1}{x^2} dx = \left. \frac{-1}{x} \right|_{x=a}^{x=b} = \frac{-1}{b} - \frac{-1}{a}$$

The above formal calculation does not work for $[a, b] = [-1, +1]$:



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Explicit Formulas for Sums

Consider the sum of the first n positive integers:

$$\begin{aligned}\sum_{k=1}^{n} k &= 1 + 2 + 3 + \cdots + n \\ &= \frac{1}{2} n(n+1).\end{aligned}$$

The geometric sum for the ratio $r \neq 1$ is

$$\begin{aligned}\sum_{k=0}^{n} r^k &= 1 + r + r^2 + \cdots + r^n \\ &= \frac{1 - r^{n+1}}{1 - r}.\end{aligned}$$

In the limit, as $r \rightarrow 1$, the sum equals $n + 1$.

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Limits of Sums

The sum with infinitely many terms

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

is equivalent to

$$\lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{k^2}$$

which equals

$$\frac{1}{6}\pi^2.$$