Linear Algebra

1. Matrices over Fields
   - systems of linear equations
   - matrix factorizations
2. Matrices over Rings
   - row reduction without divisions
   - resultants to eliminate variables

MCS 320 Lecture 30
Introduction to Symbolic Computation
Jan Verschelde, 17 July 2023

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Systems of Linear Equations

Rational, real, or complex numbers are examples of *fields*, sets of numbers for which the division operation works.

We write a linear system

$$\left\{ \begin{array}{rcl} a_{1,1}x_1 + a_{1,2}x_2 & = & b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 & = & b_2 \end{array} \right.$$

in matrix-vector notation: $A\mathbf{x} = \mathbf{b}$, with

$$A = \left[\begin{array}{cc} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{array} \right], \quad \mathbf{x} = \left[\begin{array}{c} x_1 \\ x_2 \end{array} \right], \quad \mathbf{b} = \left[\begin{array}{c} b_1 \\ b_2 \end{array} \right],$$

where

- A is *the coefficient matrix* of the linear system,
- $\mathbf{b}$ is *the right hand side vector* of the linear system.

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Matrix Factorizations

1. LU factorization, to reduce to triangular forms

   $A = LU$, L is lower triangular, U is upper triangular

2. QR factorization, for least squares solving

   $A = QR$, Q is orthogonal, R is upper triangular

3. Eigenvalue Decomposition, to reduce to a diagonal form

   $A = V \Lambda V^{-1}$, Λ is diagonal, eigenvectors in V

4. Singular Value Decomposition, for the nearest singular matrix

   $A = U^T \Sigma V$, U, V are orthogonal, Σ is diagonal

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Greatest Common Divisors

Make the second element in $\mathbf{x} = \begin{bmatrix} 810 \\ 266 \end{bmatrix}$ zero.

1. Compute the cofactors of the greatest common divisor:

$$\gcd(810, 266) = 2 = (-22) \times 810 + 67 \times 266.$$

2. Define the unimodular coordinate transformation

$$U = \begin{bmatrix} -22 & 67 \\ -266/2 & 810/2 \end{bmatrix}.$$

We have $\det(U) = 1$ and $U\mathbf{x} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

Unimodular matrices reduces integer matrices to triangular form *without divisions*.

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Resultants to Eliminate Variables

Let p and q be two polynomials in x and y, with rational coefficients:

$$p, q \in \mathbb{Q}[x, y].$$

1. View p and q as polynomials in $\mathbb{Q}[y][x]$.
2. The condition for $p, q \in \mathbb{Q}[y][x]$ to have a common factor is a polynomial in y, *the resultant of p and q with respect to x*.

The computation is reduced to a linear algebra problem:

the resultant is the determinant of the Sylvester matrix.

Application: the elimination of variables, as an alternative
to a Groebner basis with a lexicographic term order.