Follow the instructions below:

1. The exam must be solved individually.
2. Submitting materials retrieved from the internet is plagiarism.
3. You may use all notebooks posted on the course web site and your own notebooks.
4. Solutions must be in a Jupyter notebook, with a SageMath kernel.
5. The questions are provided in a Jupyter notebook.
   You may use that notebook to formulate your answers to the questions.
6. Answers must be submitted before, or at 3:40pm.
7. Submit to gradescope.
8. Not submitting any answers will by default result in a zero score.
9. During the exam no questions will be answered, so do not ask questions.

Good Luck!

1. Compute the first 10 consecutive rational approximations of $\alpha = \sin(\pi/3)$ with the consecutive convergents of the continued fraction approximation.

   Verify the accuracy of the approximations via their relative errors. /15

2. Verify that the polynomial $p = x^2 + 3x + 4$ over the finite field of five elements is irreducible.

   (a) Declare `a` as a formal root of p and extend the finite field with five elements with `a`. How many elements does this extended number field have? Justify your answer.

   (b) In this extended number field, was is the multiplicative inverse of `a + 1`? /10

3. Compute a numeric factorization of $p = x^3 + 2x + 3$.

   Expand the factorization and compare the coefficients of the expanded form with the coefficients of p. What is the sum of the errors on all coefficients? /10

4. What is the symbolic ring? Give a good application of the symbolic ring in SageMath. /10

5. Make a fast callable object of $\dfrac{\sin(x) + \cos(xy) + 2}{x + y - 1}$ and draw the expression tree. /10

6. Let $\dfrac{1}{n}\sum_{k=0}^{n-1} \exp\left(\left(\dfrac{k}{n}\right)^2\right) \approx \int_0^1 \exp(x^2)dx$

   where `s = lambda n: float(sum([exp((k/n)**2) for k in range(n)]))/n`

   is a function which defines the approximation for the integral.

   (a) Time the execution of `s` for $n = 10000$. Explain why `s` is inefficient.

   (b) Apply vectorization to improve the efficiency. Verify the correctness. Time the execution of the vectorized function for $n = 10000$, compare with timings of `s`. /15

7. Let $q = \dfrac{2x^5 - 1}{x^5 - 1}$ be a polynomial with complex coefficients.

   (a) Compute the partial fraction decomposition of q. Relate the denominators of the partial fraction decomposition to the roots of the denominator of q.

   (b) Given the partial fraction decomposition of q, reconstruct q. Describe what you observe when you normalize. /10

8. Transform $q = (u - v)^3 + \dfrac{(u + v)^5}{(u - v)^4}$ into $r = \dfrac{(u - v)^7 + (u + v)^5}{(u - v)^4}$, without typing r. /10

9. Let $p = 6x^4y^3 - x^2y^6 + x^2y^3 + x^2 - 2y^7 - y^3$.

   Without retyping p, convert p into $-x^2y^6 + 6x^4y^3 - 2y^7 + x^2y^3 - y^3 + x^2$. /10