Follow the instructions below:

1. The exam must be solved individually.
2. Submitting materials retrieved from the internet is plagiarism.
3. You may use all notebooks posted on the course web site and your own notebooks.
4. Solutions must be in a Jupyter notebook, with a SageMath kernel.
5. The questions are provided in a Jupyter notebook.
   You may use that notebook to formulate your answers to the questions.
6. Answers must be submitted before, or at 3:40pm.
7. Submit to gradescope.
8. Not submitting any answers will by default result in a zero score.
9. During the exam no questions will be answered, so do not ask questions.

Good Luck!

1. Consider $f_k(x_1, x_2, \ldots, x_n) = x_k x_{1+((k+1) \bmod n)} + \sum_{i=1}^{k} x_i$. Define a function `F` which takes on input a list `X` of variables and an index `k` in $\{1, 2, \ldots, n\}$. `F(X, k)` returns $f_k(x_1, x_2, \ldots, x_n)$. Show that your function works for $n = 8$. /10

2. Write an efficient, recursive function `P` to compute p_n defined by

$$p_0(x) = 1, \ p_1(x) = x, \ \text{for } n > 1, \ p_n(x) = (x-4)p_{n-1}(x) + 5p_{n-2}(x).$$

   The function `P` takes on input n and x and returns p_n.
   Make sure your function can compute $p_{100}(x)$. /15

3. Consider the point $P = (1, 0)$ on the curve $3y^2 - 2xy - 5x + 5 = 0$.
   Compute the slope of the tangent line to the curve at P. /10

4. Consider $f(x) = \displaystyle\int_x^5 \frac{1}{2-t} dt$.
   Use assumptions on x to evaluate $f(x)$ for all cases of x. /10

5. Consider $p = 2x^5 - 9x^4 + 6x^3 + 3x^2 - 2x + 1$.
   (a) Compute a symbolic-numeric factorization of p.
   (b) Expand the factorization and compare the coefficients of the expanded form with the coefficients of p. What is the sum of the errors on all coefficients? /10

6. Plot the curve defined by $(x(t), y(t)) = (t^3+1)/(t^2+3), (t^3+1)/(t^2+1))$, for $t \in [-1, +1]$.
   Define the system $(t^2+3)X - (t^3+1) = 0$, $(t^2+1)Y - (t^3+1) = 0$, introducing X and Y.
   Eliminate t from the system via a lexicographic Groebner basis. /15

7. Generate a random rational matrix A of dimension 10, where the numerator and denominator of each random number is bounded by 1.
   Compute the LU factorization of A and report the number in U that is largest in size.
   Repeat this experiment of a matrix B of dimension 20. Compare the growth of the numbers of the U of A and the U of B. Write one sentence to conclude your observations. /10

8. Formulate the Laplace transform of $\dfrac{d^2}{d\,t^2}y(t) + 2\dfrac{d}{d\,t}y(t) + 5y(t) = 50\cos(t)$.
   (a) Compute the general expression for the solution using the inverse Laplace transform.
   (b) In the expression for the solution, substitute the initial conditions $y(0) = 0$, $y'(0) = 0$. /10

9. Let $L = [(8,-8), (-7,-8), (-3,-4), (-7,0), (-6,3), (2,6), (-5,-2), (9,-3)]$.
   Let P be the polygon spanned by the points in L. Compute the H-representation of P.
   How many edges does P have? /10