Rational Polynomials

1. Rational Expressions and Rational Polynomials
   - arithmetical expressions and polynomials
   - automatic simplification and expression swell
2. Normal Forms
   - evaluate polynomials efficiently with the Horner form
   - the partial fraction decomposition for rational polynomials

MCS 320 Lecture 12
Introduction to Symbolic Computation
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Rational Expressions and Rational Polynomials

Consider

$$\frac{x^3 - 1}{x - 1}.$$

- As an arithmetical expression, it equals

  ```
  (x^3 - 1)/(x - 1)
  ```

  that is: x^3 minus one, divided by $x - 1$.

- As a quotient of two polynomials, it equals

  ```
  x^2 + x + 1
  ```

  that is: the sum of x^2, x, and 1, a polynomial.

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Automatic Simplification and Expression Swell

The automatic simplification of a quotient of two polynomials removes the common factors from numerator and denominator.

But this automatic simplification is not always good, consider:

$$\frac{x^{1000} - 1}{x - 1}.$$

Definition (expression swell)

In an exact calculation, *expression swell* happens
when the size of the numbers and/or expressions grow exponentially.

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the Horner Form for Efficient Polynomial Evaluation

Every polynomial in one variable of degree d can be evaluated with d multiplications and d additions.

The symbolic justification for the statement is the Horner form.

For example:

$$\begin{aligned} x^4 + 2x^3 + 3x^2 + 4x + 5 &= 5 + x(4 + x(3 + x(2 + x))) \\ &= (((x + 2)x + 3)x + 4)x + 5. \end{aligned}$$

Definition (the Horner form)

For a polynomial p in one variable x, *the Horner form* is

- p, if $\deg(p) < 2$,
- $c_0 + x\,h$, where c_0 is the constant coefficient of p, and h is the Horner form of $(p - c_0)/x$.

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the Partial Fraction Decomposition

for Rational Polynomials

Consider a quotient of two polynomials $\dfrac{p(x)}{q(x)}$.

- Let $q(x) = (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)$ with mutually distinct roots: $\alpha_i \neq \alpha_j$, for all $i \neq j$.
- Let $p(x)$ be any polynomial of degree $\deg(p) < n$.

Then the *partial fraction decomposition* is

$$\frac{p(x)}{q(x)} = \sum_{i=1}^{n} \frac{p(\alpha_i)}{q'(\alpha_i)} \left(\frac{1}{x - \alpha_i} \right).$$

Observe: $\alpha_i \neq \alpha_j$, for all $i \neq j$ implies $q'(\alpha_i) \neq 0$.