Solving Equations

1. The Apollonius Problem
   - circles touching three given circles
   - systems of polynomial equations

2. Groebner Bases
   - a lexicographic term order triangulates

MCS 320 Lecture 29
Introduction to Symbolic Computation
Jan Verschelde, 21 July 2025

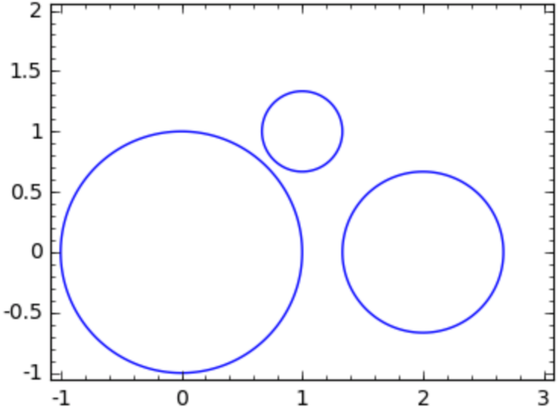
Solving Equations

1. The Apollonius Problem
   - circles touching three given circles
   - systems of polynomial equations

2. Groebner Bases
   - a lexicographic term order triangulates

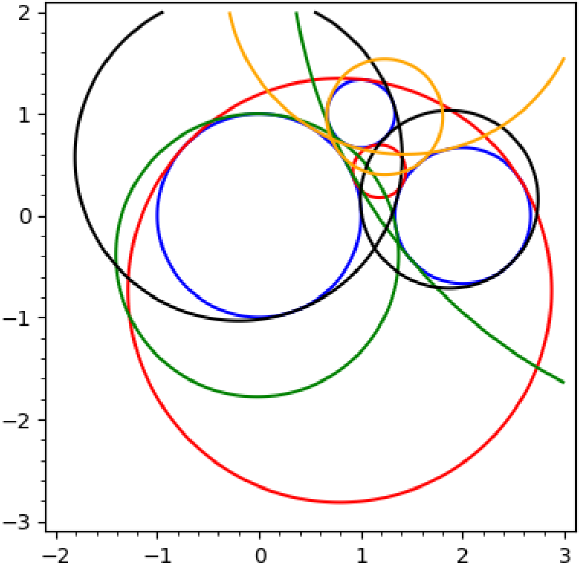
The Apollonius Problem

Given are three circles:



Compute all circles that touch these three circles.

All Circles touching the Three Given Circles



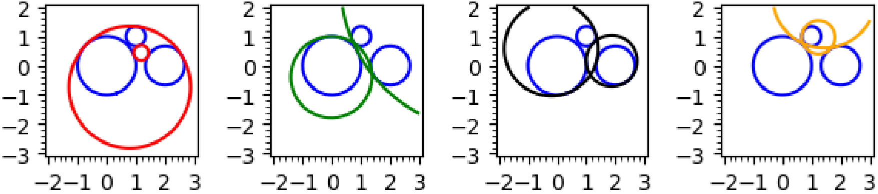
Solving Equations

1. The Apollonius Problem
   - circles touching three given circles
   - systems of polynomial equations

2. Groebner Bases
   - a lexicographic term order triangulates

All Circles touching the Three Given Circles

The touching circles are computed in pairs:



The center (x, y) and radius r of the first pair are the solutions of

$$\left\{ \begin{array}{rcl} (x - c_{1,x})^2 + (y - c_{1,y})^2 - (r - r_1)^2 & = & 0 \\ (x - c_{2,x})^2 + (y - c_{2,y})^2 - (r - r_2)^2 & = & 0 \\ (x - c_{3,x})^2 + (y - c_{3,y})^2 - (r - r_3)^2 & = & 0 \end{array} \right.$$

where the three circles are defined by the coordinates of the centers $(c_{1,x}, c_{1,y}), (c_{2,x}, c_{2,y}), (c_{3,x}, c_{3,y})$ with corresponding radii r_1, r_2, and r_3.

Solving Equations

1. The Apollonius Problem
   - circles touching three given circles
   - systems of polynomial equations

2. Groebner Bases
   - a lexicographic term order triangulates

a Lexicographic Term Order Triangulates

The polynomials in the system are ordered lexicographically.

Given a system, a Groebner basis generalizes

- the Euclidean algorithm for computing greatest common divisors of polynomials in one variable, and
- Gaussian elimination for linear systems.

For the problem of Apollonius, the three input circles are defined by the centers $(0, 0)$, $(2, 0)$, $(1, 1)$ and radii 1, 2/3, and 1/3.

The Groebner basis for the first pair of solutions is then

$$x + \frac{1}{6}r - \frac{41}{36}, \quad y + \frac{1}{2}r - \frac{11}{36}, \quad r^2 - \frac{71}{39}r - \frac{253}{468},$$

which shows there are indeed two solutions.