Substitution, Expansion, Factorization

1. Substitution
   - rewriting expressions
   - simultaneous, sequential, syntactical substitution
2. Expansion and Factorization
   - collecting terms in an expression
   - numeric, exact, and symbolic factorization

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Introduction to Symbolic Computation
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Rewriting Expressions

Rewrite

$$(x+y)^2 + \frac{1}{(x+y)^2} \quad \text{into} \quad \frac{(x+y)^4+1}{(x+y)^2}.$$

Problem: bringing on the same denominator expands the numerator.

Solution:

1. replace $x+y$ by z,
2. bring on the same denominator,
3. replace z by $x+y$.

The *substitution* of $x+y$ by z prevents the expansion.

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Simultaneous, Sequential, Syntactical Substitution

1. Replace all variables at one: *simultaneous substitution*.

$$a + 2b + 3c \;\Big|_{a=b}\;\Big|_{b=c}\;\Big|_{c=a}\; b + 2c + 3a$$

2. Replace variables one after the other: *sequential substitution*.

$$a + 2b + 3c \xrightarrow{a=b} 3b + 3c \xrightarrow{b=c} 6c \xrightarrow{c=a} 6a$$

3. Replace in the string representation: *syntactical substitution*.
   By default, substitutions respect the mathematical structure.
   With strings, we can take slices and replace any character.

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Collecting Terms in an Expression

The product

$$\left(a+b+c\right) \star \left(x^2+x+1\right)$$

is not expanded automatically because of *expression swell*.

But we may want to collect terms as follows:

$$\left(a+b+c\right) x^2 + \left(a+b+c\right) x + \left(a+b+c\right).$$

Solved by two substitutions:

1. Substitute $a+b+c$ by z before expanding and then
2. substitute z by $a+b+c$ in the expanded expression.

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Numeric, Exact, and Symbolic Factorization

Factorization is the algebraic view on root finding.

1. Numeric is over the complex numbers, is approximate.
2. Exact is over the integers, rationals, or finite field.
3. Symbolic is with the addition of formal roots, algebraic numbers.

Be mindful of the *type casting* of the coefficients of the polynomial.

1. Numerically, the coefficients are cast into the complex numbers.
2. In exact factorization, we work with the same type of coefficients.
3. Symbolically, we cast the polynomial into the extended numbers.