Symbolic-Numeric Computing

1. Interval Arithmetic
   - deciding the size of the working precision
   - symbolic-numeric factorization
2. Constrained Optimization
   - optimize subject to constraints
   - Lagrange multipliers

MCS 320 Lecture 25
Introduction to Symbolic Computation
Jan Verschelde, 10 July 2024

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How Much Precision is Needed for Accuracy?

Problem: Evaluate $f(x,y) =$

$$(333.75 - x^2)y^6 + x^2(11x^2y^2 - 121y^4 - 2) + 5.5y^8 + x/(2y)$$

at $(77617, 33096)$.

An example of Stefano Taschini: Interval Arithmetic: Python Implementation and Applications. In the *Proceedings of the 7th Python in Science Conference (SciPy 2008).*

What are the problems?

- What must the precision be for accurate results?
- Increasing the precision often gives false confidence.
- How to compute bounds on the accuracy of an approximation?

Solution: compute with intervals $[a, b]$.

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A Symbolic-Numeric Factorization

- A numeric factorization is over the complex numbers in $\mathbb{C}$.
- A symbolic factorization extends $\mathbb{Q}$ with algebraic numbers.

A *symbolic-numeric factorization* of a polynomial p uses complex interval arithmetic to represent the roots of p.

For example, in 10 bits of precision, $p = x^3 + x + 1$ factors as

$$(x - 0.341? - 1.17?I)(x - 0.341? + 1.17?I)(x + 0.682?),$$

where $I = \sqrt{-1}$ and 0.682? is short for $[0.68164, 0.68262]$.

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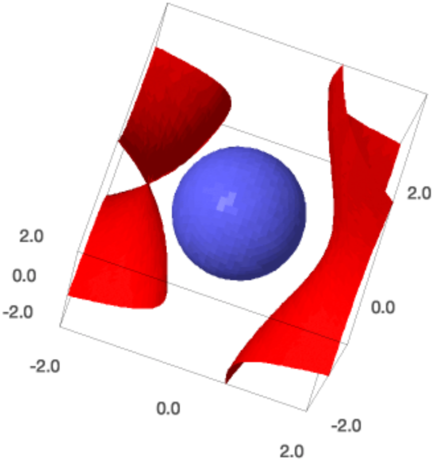
Constrained Optimization

Find the point (x, y, z) on a sphere where $f(x, y, z)$ is optimal.



The constraint is the blue sphere, the target f is drawn in red.

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Lagrange Multipliers

Let $f(x,y,z)$ be the target, what we want to optimize, and $g(x,y,z) = 0$ is the constraint.

At an optimal solution:

$$\nabla f \parallel \nabla g,$$

the gradients of f and g are parallel to each other.

The $\nabla f \parallel \nabla g$ is expressed via the multiplier λ.

Solve the polynomial system

$$\left\{ \begin{array}{rcl} \nabla f(x,y,z) - \lambda \nabla g(x,y,z) & = & 0 \\ g(x,y,z) & = & 0. \end{array} \right.$$