Computing with Functions

1. Making Expressions
   - expressions in any number of variables
   - polynomials defined by lists

2. Composing Functions
   - modeling a trajectory
   - the Newton operator

MCS 320 Lecture 21
Introduction to Symbolic Computation
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Expressions in any Number of Variables

Consider the definition of f_i in $x_1, x_2, \ldots, x_n$:

$$f_i = x_i^3 + \left(\sum_{\substack{j=1 \\ j \neq i}}^{n} (i+j+1) x_j \right) - 1, \quad i = 1, 2, \ldots, n.$$

For $n = 4$:

$$\begin{array}{rcl}
i = 1 & : & f_1 = x_1^3 + 4x_2 + 5x_3 + 6x_4 - 1, \\
i = 2 & : & f_2 = x_2^3 + 4x_1 + 6x_3 + 7x_4 - 1, \\
i = 3 & : & f_3 = x_3^3 + 5x_1 + 6x_2 + 8x_4 - 1, \\
i = 4 & : & f_4 = x_4^3 + 6x_1 + 7x_2 + 8x_3 - 1.
\end{array}$$

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Polynomials Defined by Lists

A polynomial p in n variables $x_1, x_2, \ldots, x_n$ of m terms is defined by

1. a list of m coefficients, $c_1, c_2, \ldots, c_m$, and
2. a list of m exponent tuples $e_1, e_2, \ldots, e_m$, where

$$e_i = (e_{i,1}, e_{i,2}, \ldots, e_{i,n}), \quad i = 1, 2, \ldots, m.$$

In short, the polynomial is then defined by:

$$p = \sum_{i=1}^{m} c_i \prod_{j=1}^{n} x_j^{e_{i,j}}$$

The $\sum$ and $\prod$ are *reduction operators on lists*.

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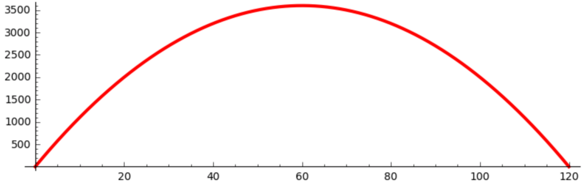
Modeling a Trajectory — separating space from time

Consider modeling the trajectory of a projectile by a parabola:

1. at time $t = 0$, the projectile is launched from the ground,
2. at time $t = 45$, it hits the ground 120 miles further.

First we model the shape: $y(x) = x(120 - x)$ is a parabola.



Second, assuming constant horizontal speed: $x(t) = 120/45t$.

Then our model is defined by the *composition of functions*

$$f(t) = y(x(t)).$$

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the Newton operator

To solve $f(x) = 0$, we define *the Newton operator*

$$N(f, x) = x - \frac{f(x)}{f'(x)}.$$

Starting at $x = x_0$, three steps are performed by

$$x_3 = N(N(N(x_0))).$$

Example: $f(x) = x^2 - 2$.

Running the Newton operator, starting at $x_0 = 2$, results in increasingly more accurate rational approximations for $\sqrt{2}$.

For regular problems, we observe quadratic convergence:

Newton's method doubles the accuracy in each step.