

## 27. Differential Equations

In a crude overview, we can say that Maple offers three different types of solutions to ODEs:

1. analytic solutions, in the form of an explicit formula for the solution;
2. the solution expressed as a series; and
3. a numerical solution to the ODE.

While an analytic solution may seem the most desirable solution, we must know that (similar to integration where formal antiderivatives do not always exist) a nice formula for the solution may not exist.

### 27.1 Analytic Solutions

If we have no preference for the type of method, we just call `dsolve` without any special parameters. We define `y` as a function of `x` by the following differential equation:

```
[> ODE := x*diff(y(x),x) = y(x)*ln(x*y(x)) - y(x);
[> s := dsolve(ODE,y(x));
```

Maple returns a formal solution where `_C1` is any arbitrary complex constant. How can we verify the solution?

```
[> yx := rhs(s);
[> syx := subs(y(x)=yx,ODE);
[> simplify(lhs(syx));
[> simplify(rhs(syx));
```

### 27.2 Taylor Series Methods

With this method, the solution is developed as a Taylor series and we use the conditions imposed by the ODE to find values for the coefficients of the Taylor series expansion.

Let us look at the pendulum, defined by a second-order differential equation :

```
[> diffeq := l*diff(theta(t),t$2) = -g*sin(theta(t));
```

The `l` in the equation is the length of the pendulum, `theta(t)` is the angle measuring the deviation from its vertical position and `g` is the gravitational constant. Since this is a second-order equation, we need two initial conditions. At time `t=0`, the pendulum is in vertical position, so `theta(0) = 0`, and its velocity is some `v[0]`.

```
[> inits := theta(0) = 0, D(theta)(0) = v[0];
```

The ODE is thus defined by a differential equation and two initial conditions:

```
[> ODE := {diffeq,inits};
```

Observe the very important difference in the use of **diff** and **D**. In the equation, we set up a formula and use **diff**. In the initial condition, we regard `theta` as a function of time and must use **D**.

```
[> sol := dsolve(ODE,theta(t),series);
```

To crank up the order of approximation, we change the variable `Order`. See the lecture on series to turn this into a function.

## 27.3 Numerical Solutions

Using numerical algorithms, we can study more advanced oscillators than just the pendulum. Let us take the van der Pol equation:  $y'' - (1 - y^2) * y' + y = 0$ , with  $y(0) = 0, y'(0) = -0.1$ .

```
[> diffeq := diff(y(t),t$2) - (1-y(t)^2)*diff(y(t),t) + y(t) = 0;
[> inits := y(0) = 0, D(y)(0) = -0.1;
[> ODE := {diffeq,inits};
[> sol := dsolve(ODE,y(t),type=numeric);
[> sol(0.1);
```

We see that we get a list of three elements when we evaluate the solution at any given time. But if we are only interested in the evolution of  $y(t)$  in function of time, we need to select from the output of dsolve the right function for plotting purposes:

```
[> solfun := t -> rhs(sol(t)[2]);
[> solfun(0.1);
[> plot(solfun,0..20);
```

## 27.4 DEtools

The package DEtools offers a wide range of graphical tools to study ODEs:

```
[> with(DEtools);
```

Here we take the van der Pol equation again and make a “phase portrait” of two curves defined by the equation. We first transform our second-order equation into a system of two first-order linear equations:

```
[> ODEs := diff(y(t),t) = yy(t), diff(yy(t),t) - (1-y(t)^2)*yy(t) + y(t)=0;
```

If we want to see two trajectories, we cook up two sequences of initial conditions:

```
[> inits1 := y(0) = 0, yy(0) = -0.1;
[> inits2 := y(0) = 0.1, yy(0) = 0;
[> DEplot({ODEs},[y(t),yy(t)],t=0..10,[[inits1],[inits2]]);
```

## 27.5 Assignments

1. Show that the solution  $y(x)$  to the equation

$$\sqrt{a^2 - y^2} - a \ln \left( a + \sqrt{a^2 + y^2} \right) + a \ln(y) + x = C$$

is a solution of the differential equation

$$y' = -\frac{y}{\sqrt{a^2 - y^2}}.$$

2. Solve the following ODE:

$$3y^2 y' + 16x = 12xy^2$$

and verify the solution you find.

3. Solve the following initial value problem:

$$y'' - y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$