9. Linear Programming in MATLAB

One of the most widespread commercial applications of scientific computation is linear programming.

9.1 An optimization problem

Suppose a farmer has 75 acres to plant and must decide how much farm land to devote to crop $x$ or crop $y$. Crop $x$ brings in more revenue than $y$, as can be seen from the profit function $P(x, y)$:

$$P(x, y) = 143x + 60y.$$  

In maximizing the profit $P(x, y)$, all should be devoted to crop $x$, but farming is not that simple. There are additional constraints (other than $x + y \leq 75$). For instance, crop $x$ requires more storage space than crop $y$, and if the total storage is limited to 4,000, then we have to take the following constraint

$$110x + 30y \leq 4,000$$

into account. The crops just do not grow for free. As the farmer cannot spend more than 15,000 to grow the crops, the third constraint takes the form

$$120x + 210y \leq 15,000.$$ 

In addition to the trivial constraints $x \geq 0$ and $y \geq 0$, our problem is summarized into:

$$\begin{align*}
\max_{x,y} & \quad 143x + 60y \\
\text{subject to} & \quad x + y \leq 75 \\
& \quad 110x + 30y \leq 4,000 \\
& \quad 120x + 210y \leq 15,000 \\
& \quad x \geq 0, y \geq 0
\end{align*}$$

Since there are only two unknowns involved, we may graph all constraints:

```matlab
>> x = 0 : 80; % range for graph
>> y1 = max(75 - x, 0); % x + y \leq 75 farm land
>> y2 = max((4000 - 110 * x)/30, 0); % 110x + 30y \leq 4000 storage
>> y3 = max((15000 - 120 * x)/210, 0); % 120x + 210y \leq 15000 expenses
>> ytop = min([y1; y2; y3]); % array of minima
>> area(x, ytop); % filled area plot
```

The shaded area enclosed by the constraints is called the feasible region, which is the set of points satisfying all the constraints. If this region is empty, then the problem is said to be infeasible, and it has no solution.

The lines of equal profit $p$ are given by $p = 143x + 60y$. If we fix $p$ to, say 50, then all points $(x, y)$ which satisfy $143x + 60y$ yield the same profit 50.

```matlab
>> hold on; % range for graph
>> [u v] = meshgrid(0 : 80, 0 : 80); % x + y \leq 75 farm land
>> contour(u, v, 143 * u + 60 * v); % 110x + 30y \leq 4000 storage
>> hold off; % filled area plot
```

To find the optimal solution, we look at the lines of equal profit to find the corner of the feasible region which yields the highest profit. This corner can be found at the farthest line of equal profit which still touches the feasible region.
9.2 The command linprog

The command linprog from the optimization toolbox implements the simplex algorithm to solve a linear programming problem in the form

\[
\begin{align*}
\min_{x} & \quad f^T x \\
\text{subject to} & \quad Ax \leq b
\end{align*}
\]

where \( f \) is any vector and the matrix \( A \) and vector \( b \) define the linear constraints. So our original problem is translated into the format

\[
\begin{align*}
\max & \quad 143x + 60y \\
\text{subject to} & \quad 
\begin{cases}
  x + y \leq 75 \\
  110x + 30y \leq 4000 \\
  120x + 210y \leq 15000
\end{cases}
\end{align*}
\]

Observe the switching of signs to turn the max into a min and to deal with the \( \geq \) constraints. Now we are ready to solve the problem. First we set up the vectors and matrices:

\[
\begin{align*}
& >> f = [-143 \ -60] \\
& >> A = [120 \ 210; 110 \ 30; 1 \ -1 \ 0 \ 0 \ -1] \\
& >> b = [15000; 4000; 75; 0; 0]
\end{align*}
\]

The optimization toolbox has the command linprog:

\[
\begin{align*}
& >> \text{linprog}(f,A,b) \quad \text{% optimize} \\
& >> -f * \text{ans} \quad \text{% compute profit}
\end{align*}
\]

The latest versions of MATLAB have the command simlp, which is very much like linprog.

9.3 Assignments

1. Suppose General Motors makes a profit of $100 on each Chevrolet, $200 on each Buick, and $400 on each Cadillac. These cars get 20, 17, and 14 miles a gallon respectively, and it takes respectively 1, 2, and 3 minutes to assemble one Chevrolet, one Buick, and one Cadillac. Assume the company is mandated by the government that the average car has a fuel efficiency of at least 18 miles a gallon. Under these constraints, determine the optimal number of cars, maximizing the profit, which can be assembled in one 8-hour day. Give all MATLAB commands and the final result.

2. Consider the following optimization problem:

\[
\begin{align*}
\min_{x,y} & \quad 2x + y \\
\text{subject to} & \quad 
\begin{cases}
  x + y \geq 4 \\
  x + 3y \geq 12 \\
  x - y \geq 0 \\
  x \geq 0, \ y \geq 0
\end{cases}
\end{align*}
\]

(a) Use MATLAB to graph the constraints.

(b) Use linprog to compute the solution.