Maple Lecture 27. Differential Equations

In a crude overview, we can say that Maple offers three different types of solutions to ODEs:

1. analytic solutions, in the form of an explicit formula for the solution;
2. the solution expressed as a series; and
3. a numerical solution to the ODE.

While an analytic solution may seem the most desirable solution, we must know that (similar to integration where formal antiderivatives do not always exist) a nice formula for the solution may not exist.

The command `dsolve` is one of the most useful solvers in Maple as differential equations have lots of applications, see [1]. This lecture is a selection of the material in [2, Chapter 17].

27.1 Analytic Solutions

If we have no preference for the type of method, we just call `dsolve` without any special parameters. We define \( y \) as a function of \( x \) by the following differential equation:

\[
\text{ODE} := x \frac{\text{diff}(y(x),x)}{} = y(x) \ln(x*y(x)) - y(x);
\]

\[
\text{s} := \text{dsolve}(\text{ODE}, y(x));
\]

Maple returns a formal solution where \( C_1 \) is any arbitrary complex constant. How can we verify the solution?

\[
\text{yx} := \text{rhs(s)};
\]

\[
\text{syx} := \text{subs}(y(x)=yx,\text{ODE});
\]

\[
\text{simplify(lhs(syx))};
\]

\[
\text{simplify(rhs(syx))};
\]

27.2 Taylor Series Methods

With this method, the solution is developed as a Taylor series and we use the conditions imposed by the ODE to find values for the coefficients of the Taylor series expansion.

Let us look at the pendulum, defined by a second-order differential equation:

\[
\text{diffeq} := l \frac{\text{diff}^2(\theta(t),t)}{} = -g \sin(\theta(t));
\]

The \( l \) in the equation is the length of the pendulum, \( \theta(t) \) is the angle measuring the deviation from its vertical position and \( g \) is the gravitational constant. Since this is a second-order equation, we need two initial conditions. At time \( t=0 \), the pendulum is in vertical position, so \( \theta(0) = 0 \), and its velocity is some \( v[0] \).

\[
\text{inits} := \theta(0) = 0, \ D(\theta)(0) = v[0];
\]

The ODE is thus defined by a differential equation and two initial conditions:

\[
\text{ODE} := \{\text{diffeq,inits}\};
\]

Observe the very important difference in the use of `diff` and \( D \). In the equation, we set up a formula and use `diff`. In the initial condition, we regard \( \theta \) as a function of time and must use \( D \).

\[
\text{sol} := \text{dsolve}(\text{ODE,theta}(t),\text{series});
\]

To crank up the order of approximation, we change the variable `Order`. See the lecture on series to turn this into a function.
27.3 Numerical Solutions

Using numerical algorithms, we can study more advanced oscillators than just the pendulum. Let us take the van der Pol equation:

\[ y'' - (1 - y^2) y' + y = 0, \] with \( y(0) = 0, y'(0) = -0.1. \)

```maple
> diffeq := diff(y(t),t$2) - (1-y(t)^2)*diff(y(t),t) + y(t) = 0;
> inits := y(0) = 0, D(y)(0) = -0.1;
> ODE := {diffeq,inits};
> sol := dsolve(ODE,y(t),type=numeric);
> sol(0.1);
```

We see that we get a list of three elements when we evaluate the solution at any given time. But if we are only interested in the evolution of \( y(t) \) in function of time, we need to select from the output of dsolve the right function for plotting purposes:

```maple
> solfun := t -> rhs(sol(t)[2]);
> solfun(0.1);
> plot(solfun,0..20);
```

27.4 DEtools

The package DEtools offers a wide range of graphical tools to study ODEs:

```maple
> with(DEtools);
```

Here we take the van der Pol equation again and make a “phase portrait” of two curves defined by the equation. We first transform our second-order equation into a system of two first-order linear equations:

```maple
> ODEs := diff(y(t),t) = yy(t), diff(yy(t),t) - (1-y(t)^2)*yy(t) + y(t)=0;
```

If we want to see two trajectories, we cook up two sequences of initial conditions:

```maple
> inits1 := y(0) = 0, yy(0) = -0.1;
> inits2 := y(0) = 0.1, yy(0) = 0;
> DEplot({ODEs},{y(t),yy(t)},t=0..10,[inits1],[inits2]);
```
27.5 Assignments

1. Show that the solution \( y(x) \) to the equation
\[
\sqrt{a^2 - y^2} - a \ln\left(a + \sqrt{a^2 + y^2}\right) + a \ln(y) + x = C
\]
is a solution of the differential equation
\[
y' = -\frac{y}{\sqrt{a^2 - y^2}}
\]

2. Solve the following ODE:
\[
3y^2y' + 16x = 12xy^2
\]
and verify the solution you find.

3. Solve the following initial value problem:
\[
y'' - y = 0, \quad y(0) = 1, \quad y'(0) = 0.
\]

4. The Chebyshev polynomials \( T_n(x) \) satisfy the following differential equation:
\[
(1 - x^2)T''_n(x) - xT'_n(x) + n^2T_n(x) = 0.
\]
(a) Apply \texttt{dsolve} to the differential equation and interpret what Maple returns to you.
Note: \( T_n(x) = \cos(n \arccos(x)) \).
(b) The command \texttt{orthopoly[T]}(25,x) returns \( T_{25}(x) \).
Verify that \texttt{orthopoly[T]}(25,x) satisfies the differential equation.

5. Consider the following system of differential equations:
\[
\begin{align*}
\frac{dr}{dt} &= 1.1r(t) - 0.5r(t)f(t) \\
\frac{df}{dt} &= -0.75f(t) + 0.25r(t)f(t)
\end{align*}
\]
r(0) = 3, f(0) = 2.
This system can be used to model the evolution in time \( t \) of two species, with \( f(t) \) the size of the predator population (\( f = \text{fox} \)) and \( r(t) \) the size of the prey population (\( r = \text{rabbit} \)).

(a) Use \texttt{dsolve} to define two numerical procedures \texttt{rabbits} and \texttt{foxes} which are functions of \( t \): \texttt{rabbits}(t) = r(t) and \texttt{foxes}(t) = f(t), i.e.: \texttt{rabbits} is the solution for \( r(t) \) and \texttt{foxes} is the solution for \( f(t) \), as defined by the given system of differential equations.
(b) With these two functions, make a plot of the solution trajectories, for \( t \) going from 0 to 8.

References
