Maple Lecture 12. Representation of Expressions

In this lecture we take a detailed look at the internal representations of expressions. The material in this lecture is inspired on [1, Section 6.1 and 6.2]. See [2] for more specific details. The use of directed acyclic graphs to represent expressions has implications on the effect of a substitution, one of the major tools to manipulate expressions.

12.1 Internal representation of polynomials

There are three levels of selecting data from a polynomial. At the highest level we have \texttt{coeffs} and \texttt{coeff}. The commands \texttt{nops} and \texttt{op} work at a lower level and allow us to draw the expression tree of the polynomial. Finally, at the lowest level, we use \texttt{dismantle} to get to the Directed Acyclic Graph representation of a polynomial.

This Directed Acyclic Graph representation is motivated by the memory efficiency of Maple: every object is stored only once. So leaves in the expression tree with the same content are stored as the same end node in a directed acyclic graph. But also the substitution command can be executed more efficiently: the replacement of the same symbol (or number) in an expression occurs only once, even though the symbol (or number) may appear in several places in the expression.

```maple
> p := x^10 + 2*x^4 + 9;
> coeffs(p); # sequence of coefficients
> whattype(p); # same as op(0,p);
>
```

While \( p \) is a polynomial, it is also a general expression of type + on which the usual selectors work:

```maple
> nops(p); # number of terms
> op(p); # sequence of terms
> op(1,p); # select first term of \( p \)
> op([2,1],p); # select 1st term of 2nd term of \( p \)
>
```

With the \texttt{op} command, we can construct the expression tree, shown in Figure 1.

![Expression tree of \( x^{10} + 2x^4 + 9 \).](image)

Figure 1: Expression tree of \( x^{10} + 2x^4 + 9 \).

The expression tree is still very much a logical view on the polynomial. Internally, Maple stores an expression as a \textit{Directed Acyclic Graph}. From the output of \texttt{dismantle} we draw this graph in Figure 2.

```maple
> dismantle(p);
```

Notice: in storing the terms in the sum, the monomials come first, followed by the coefficients. So the polynomial \( x^{10} + 2x^4 + 9 \) is actually stored like \( x^{10} \times 1 + x^4 \times 2 + 9 \times 1 \).

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12.2 The side effects of a substitution

Now we understand the structure, and knowing that Maple stores every object only once, we can explain the effects of a substitution like this (see Figure 3):

```maple
> q := subs(1=Pi,p);
> dismantle(q);
```

Observe that protected is an attribute to the name Pi. When a variable has no attribute (like x), then this field of the name simply points to NULL.

Since the polynomial \( p = x^{10} + 2x^4 + 9 \) is stored like \( x^{10} \times 1 + x^4 \times 2 + 9 \times 1 \) and every object is stored only once, we now see why the command `subs(1=Pi,p)` creates the polynomial \( x^{10} \times \pi + x^4 \times 2 + 9 \times \pi. \)

12.3 Generalized rational expressions

Maple can work with polynomials where the variable is replaced by, for example a cosine:

```maple
> pcos := subs(x=cos(x),p);
> whattype(pcos); type(pcos,polynom);
```

While the expression is no longer a true polynomial, we can still ask for the coefficients :

```maple
> coeffs(pcos); 
```

But we can no longer factor the polynomial directly:
We have to work indirectly, first factor $p$ and then substitute $x$ by $fp$ in the factored polynomial:


general rational expressions are rational polynomial in disguise:

It is interesting to see the internal structure (see Figure 4):

The dismantle procedure is one of the tools in the “hackware package” of Maple. Like we could do in a low-level programming language as C, we can manipulate addresses. For example:

\[
\text{\texttt{dismantle[dec]}(p)}; \quad \text{\texttt{addressof}(p)};
\]

\[
\text{\texttt{# see the addresses in decimal notation}} \quad \text{\texttt{# verify the address of } p}
\]
12.4 Assignments

1. For the polynomial \( p = x^3 - 8x + 1 \), give the Maple command (only one command!) to select the coefficient \(-8\). Draw the expression tree for \( p \) and the directed acyclic graph of its internal representation.

2. Use dismantle to investigate the internal representation of the polynomial \( 2x(y^2 + 1)^2 \). From the output of dismantle, draw the directed acyclic graph of this polynomial.

3. Consider the following instructions

   \[
   \begin{align*}
   &> x^2 - x + 1/x - 1/x^2; \\
   &> \text{subs}\left(-1=1,%\right);
   \end{align*}
   \]

   Explain the result of this substitution.

4. Compare the internal representations of \( p = x^8 + 2x^4 + 3 \) with \( q = (x^4 + 2)x^4 + 3 \).
   Draw the directed acyclic graphs of \( p \) and \( q \) based on the output of dismantle.

5. Do \texttt{dismantle[dec]}(\texttt{p}); after \( p := x^8 + 2x^4 + 3 \).
   Explain how you can see from the output of dismantle that the name \( x \) is represented only once.

References
