Second Midterm Exam MCS 320 Jan Verschelde Friday 1 April 2016

The exam is open book, open notes and open computer.
All your answers to the questions below must be *hand written*, submitted on paper.

1. Consider the formula $\pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right)$.

   Make a function `f(k)` that returns the value of what is in the $(\cdot)$ in the sum for π. Use this function in a list comprehension to compute the first 100 hexadecimal digits of π. How accurate is this approximation for π? Write the magnitude of the error.

2. Hermite polynomials $H_n(x)$ satisfy the following recursion relation:

   $$H_{n+1}(x) = 2xH_n(x) - H'_n(x), \quad H_0(x) = 1,$$

   where $H'_n(x)$ is the derivative of $H_n(x)$ with respect to x.

   Use this recursive relation to define an efficient Sage function to compute $H_n(x)$. Make sure your function can compute $H_{100}(x)$.

3. Consider the cissoid of Diocles as defined by $x^3 - (2-x)y^2 = 0$.
   What is the slope of the tangent line at $(1,1)$?
   Write all commands you used to compute this slope.

4. Consider the function $f(x) = 1 - 10x + 0.01e^x$.
   Use interval arithmetic to show that $f(x) = 0$ has a solution in the interval $[8,9]$.

5. Consider the surface defined by the equation $f(x,y,z) = x^3 - y^2 - z^2 = 0$.

   (a) Give the Sage command to plot this surface, for $x \in [0,1]$ and for $y, z \in [-1,1]$. What do you see at (0,0,0)? Describe.

   (b) Replace z by zero and transform the curve defined by $f(x,y,0) = 0$ into polar coordinates. Write the Sage command to graph the curve in polar coordinates. Use good bounds for the range of t.

6. Let $g(x,y) = x^4 - 2y^3 + x^2y$ and consider the point $(1,0)$.

   Suppose we want to compute the point on the curve defined by $g(x,y) = 0$ that is closest to $(1,0)$. Write all commands to set up the polynomial system that you need to solve to compute this closest point.

7. Let the n-dimensional matrix A have zeros on its diagonal.
   For $i \neq j$, the (i,j)-th element of A is $+1$ if $i+j$ is even and -1 if $i+j$ is odd.

   (a) Give the Sage commands to define A for $n = 5$.

   (b) Is this A invertible? Over the integers or the rationals? Explain.
   Write the command(s) used to obtain your answer.