MCS 320 Project One : Continued Fractions
due Friday 28 June 2024, at 2pm.

When expanding rational numbers such as 1/3 in the decimal basis, we end up with the infinite sequence 0.333.... In the binary number system 1/10 has no finite expansion and the binary expansion therefore leads to a representation error. Continued fractions are an alternative way to represent rational numbers is a unique way, independent of the basis, as a finite sequence of integers, without representation errors.

In this project, we use SageMath to explore properties of continued fractions. The project description is derived from a Jupyter notebook with SageMath kernel. The best way to start the project is to download and to execute the notebook.

0. Continued Fractions of Rational Numbers

We consider nonnegative rational numbers with numerator and denominator bounded by a given size, in the code below the size is one billion.

```
size = 10^9
q = abs(QQ.random_element(num_bound=size, den_bound=size))
cq = q.continued_fraction()
```

The output `cq` is the list of convergents in the continued fraction of `q`. To display the continued fraction nicely, we execute `show(cq)`.

For rational numbers, the continued fractions do not continue forever, but stop. Conversely, any finite list of integers can be converted into a continued fraction and then evaluated into a rational number, as illustrated by the code below, which continues the sequence of above instructions.

```
Lcq = list(cq)
cLcq = continued_fraction(Lcq)
show(cLcq)
```

Then `QQ(cLcq)` will return the original number `q`.

1. The Greatest Common Divisor

The convergents appear as the quotients when running Euclid's algorithm to compute the greatest common divisor, which can be verified by printing out the quotient computed by Euclid's algorithm. So, Euclid's algorithm gives us an immediate algorithm to compute continued fractions. We now also understand that for rational numbers, the numbers in the continued fractions are bounded by the size of their numerators and denominators.

If the convergents of the continued fractions are the quotients computed by Euclid's algorithm, how can we compute the remainders?

Assignment One.

Observing the correspondence between the last convergent and the last remainder, compute the rational number computed by selecting the last convergents.

Illustrate that the numerators of the rational numbers of the last convergents give the remainders computed by Euclid's algorithm.

2. The Fibonacci Numbers

What is the rational number of the continued fraction where all numbers in the list of convergents are equal to one?

The sequence of Fibonacci numbers f_n is defined as follows: $f_0 = 0$, $f_1 = 1$, and for $n > 1$: $f_n = f_{n-1} + f_{n-2}$.

Assignment Two. Show that the list of convergents of the $(n+1)$-th Fibonacci number divided by the n-th one has length $n-1$. Run the demonstration for the first 20 Fibonacci numbers.

3. The Golden Ratio

If we consider the infinite sequence of ones in the list of integers to define the convergents, then what does this number converge to?

Assignment Three.

Evaluate the first 100 convergents and compute the decimal expansion in a real field with 200 bits of precision.

The golden ratio is $\frac{1+\sqrt{5}}{2}$.

Compute the error of the approximation obtained with the first 100 convergents, as its difference with the golden ratio. How many decimal places are correct?

4. The Deadline is Friday 28 June, at 2pm.

Upload your answer to gradescope at the latest on Friday 28 June, before 2pm.

The solution consists of one single notebook, organized according to the assignments. Mark the start of each solution to an assignment using a heading in a cell. Your notebook should run from top to bottom as a program without errors. Apply proper formatting in your notebook so it reads like a technical report if you would print it. Document the execution cells with complete sentences, properly formatted in markdown cells.

You may (not must) work in pairs for this project. A pair consists of two, not three or more. If you decide to work in a pair, then you must send me an email with the name of your partner and with the email address of your partner in the copy of the email, before 2pm on Friday 21 June. If working in a pair, then only one Jupyter notebook should be submitted.

If you have questions, concerns, or difficulties, feel free to contact me for help.

References

[1] Donald E. Knuth. *The Art of Computer Programming. Volume 2. Seminumerical Algorithms.* Third Edition, Addison-Wesley, 1998.