MCS 320 Project Three : Population Models
due Wednesday 31 July 2024, at 2pm.

In this project, we use SageMath to model population growth with ordinary differential equations. We compute symbolic and numeric solutions, and plot solution trajectories.

1. Growth Rate and Carrying Capacity

A model for population growth uses the following two parameters:

1. r is the birth rate, and
2. K is the carrying capacity.

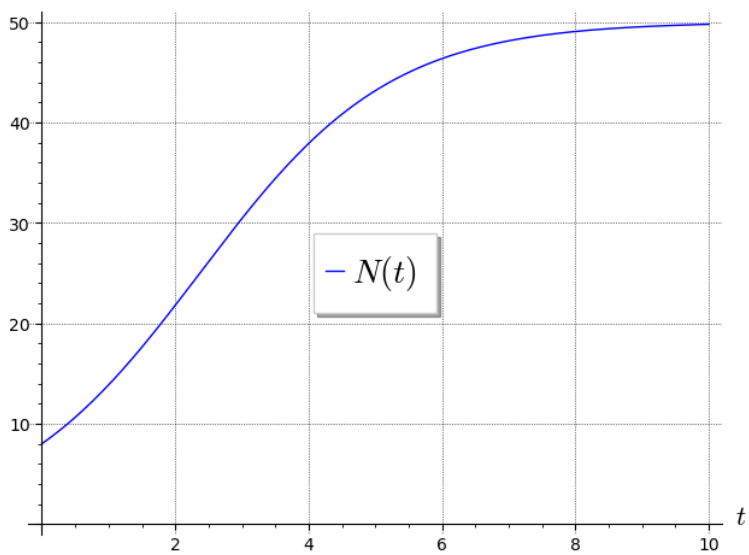
Then $N(t)$ is the population number in function of time t, as solution of the ODE:

$$\frac{d}{d\,t}N(t) = rN(t)\left(1 - \frac{N(t)}{K}\right).$$

If $N_0 = N(0)$ is the initial population size, then the symbolic solution is

$$N(t) = \frac{KN_0\exp(rt)}{N_0\exp(rt) + K - N_0}.$$

Below is the plot for $r = 0.7$, $K = 50$, $N_0 = 8$, for t from 0 to 10.



Assignment One. For the population model described above, do the following:

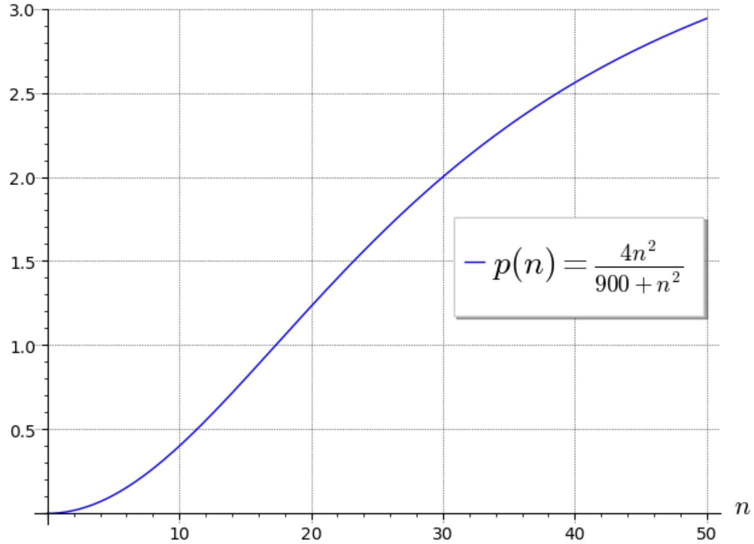
1. Compute the symbolic solution.
2. Use the symbolic solution to make the above plot.

2. Growth with Predation

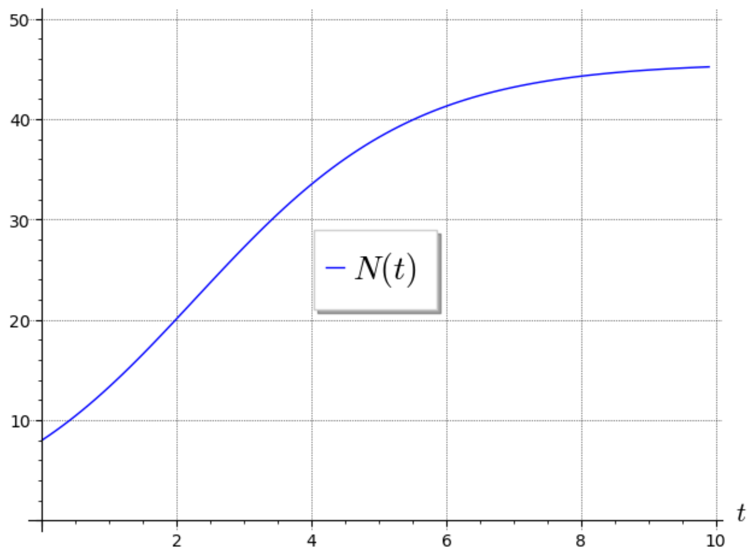
Suppose the population we model is subject to a predator. The predator is modeled by adding a function P of $N(t)$ to the differential equation:

$$\frac{d}{dt}N(t) = rN(t)\left(1 - \frac{N(t)}{K}\right) - P(N(t)).$$

A good function for P is $p(n) = \dfrac{Bn^2}{A^2 + n^2}$. For $A = 30$ and $B = 4$, the shape of $p(n)$ is shown below:



and the population with this predation term is shown below:



Assignment Two. Consider the population model with the predation term.

1. Solve the problem symbolically and describe the output.
2. Solve the problem numerically, with $N_0 = 8$, $r = 0.7$, $K = 50$, $A = 30$, and $B = 4$, for t from 0 to 10. Plot the solution.

3. Steady State Solutions

When the right hand side of the ODE equals zero:

$$rN(t)\left(1-\frac{N(t)}{K}\right)-\frac{BN(t)^2}{A^2+N(t)^2}=0,$$

then $\dfrac{d}{dt}N(t)=0$ and then the solution remains invariant over all time.

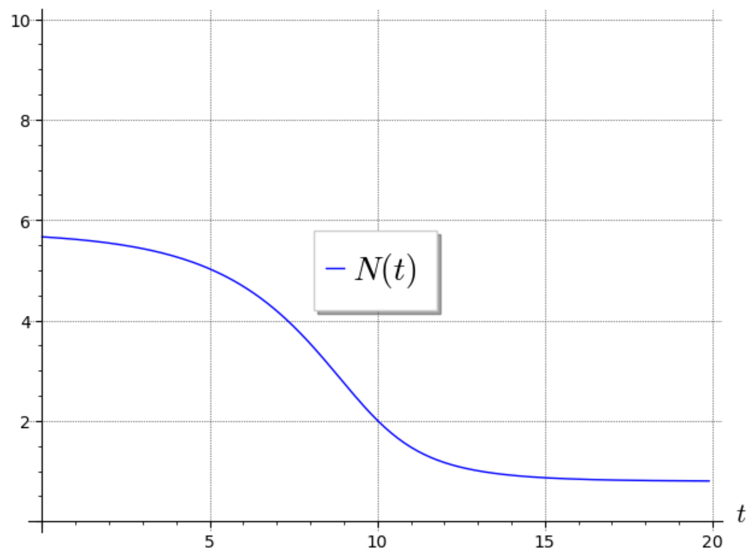
Assignment Three. Let us examine the three steady state solutions for this problem, excluding the trivial zero solution. Observe that, when clearing denominators, the right hand side of the ODE becomes a cubic equation in $N(t)$, after removing the factor which corresponds to the trivial zero solution.

1. Compute the steady state solutions for the problem, for the numerical values of the parameters: $r=0.7$, $K=50$, $A=2$, and $B=3$.

2. If N^* is a steady state solution, then let $N_0 = N^* \pm 0.1$ and plot the solution trajectories, for t from 0 to 20.

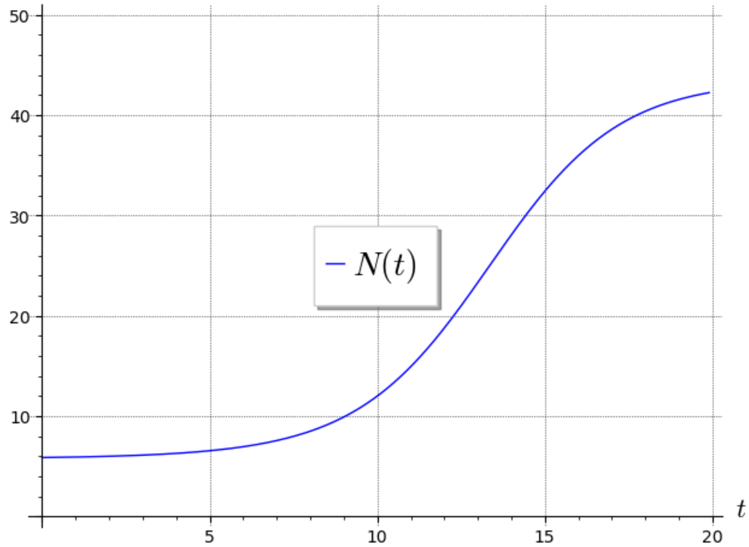
   What do you observe about the three steady state solutions?

For the middle value of N^*, we have that the solution trajectory converges to the smallest value for N^*, when $N_0 = N^* - 0.1$.



We see that the solution does not converge to the middle value of N^*, even though the N_0 was taken close enough, at $N_0 = N^* - 0.1$.

When $N_0 = N^* + 0.1$, the solution trajectory converges to the largest value for N^*.



We see that the solution does not converge to the middle value of N^*, even though the N_0 was taken close enough, at $N_0 = N^* + 0.1$.

Of the three steady state solutions, the small and the large steady state solutions are stable. The middle steady state is unstable.

4. The Deadline is Wednesday 31 July, at 2pm

Upload your answer to gradescope at the latest on Wednesday 31 July, before 2pm.

The solution consists of one single notebook, organized according to the assignments. Mark the start of each solution to an assignment using a heading in a cell. Your notebook should run from top to bottom as a program without errors. Apply proper formatting in your notebook so it reads like a technical report if you would print it. Document the execution cells with complete sentences, properly formatted in markdown cells.

You may (not must) work in pairs for this project. A pair consists of two, not three or more. If you decide to work in a pair, then you must send me an email with the name of your partner and with the email address of your partner in the copy of the email, before 2pm on Friday 26 July. If working in a pair, then only one Jupyter notebook should be submitted.

If you have questions, concerns, or difficulties, feel free to contact me for help.