Follow the instructions below:

1. The exam must be solved individually.
2. Submitting materials retrieved from the internet is plagiarism.
3. You may use all notebooks posted on the course web site and your own notebooks.
4. Solutions must be in a Jupyter notebook, with a SageMath kernel.
5. The questions are provided in a Jupyter notebook.
   You may use that notebook to formulate your answers to the questions.
6. Answers must be submitted before, or at 3:40pm.
7. Submit to gradescope.
8. Not submitting any answers will by default result in a zero score.
9. During the exam no questions will be answered, so do not ask questions.

*Good Luck!*

1. Let $N = \cos(\pi/5)$.

   (a) Compute a nearby rational approximation for $N$, with the denominator bounded by 999.

   (b) What is the accuracy of your nearby rational approximation? /15

2. Compute the square roots of $c = 5 - 4I$.

   Verify that the square of your computed roots equals $c$. /10

3. Let $p = x^3 + x + 4$ be a polynomial with coefficients in a finite field with 11 elements. Show that $p$ is irreducible.

   Declare $\alpha$ as a formal root of $p$ and show that $p$ factors over the field extended with $\alpha$. /15

4. Define an equation `eqn` that shows `cos(pi/2) == 0` when printed.

   Without retyping `eqn`, change `eqn` so `print(eqn)` shows `0 == 0`. /10

5. Consider the evaluation of $p = x^8 - 2x^7 + x^6 + 3x^5 - x^4 + 4x^3 + x + 6$.

   What is the fastest way to evaluate $p$ at hardware floats? Justify your answer. /10

6. The `f = lambda n: float(sum([(1+k/n)**2*ln(1+k/n) for k in range(1,n)])/n)` computes the right hand side of

$$\int_1^2 x^2 \ln(x) dx \approx \frac{1}{n} \sum_{k=1}^{n-1} \left(1 + \frac{k}{n}\right) \ln\left(1 + \frac{k}{n}\right).$$

   (a) Time the execution of `f` for $n = 10000$. Explain why `f` is inefficient.

   (b) Apply vectorization to improve the efficiency. Verify the correctness.
   Time the execution of the vectorized function for $n = 10000$, compare with timings of `f`. /10

7. Consider the rational expression $\dfrac{u^{1000} - v^{1000}}{u - v}$.

   Explain why it is bad to simplify this expression automatically. /10

8. Transform $q = (x-y)^2 + \dfrac{(x+y)^7}{(x-y)^2}$ into $r = \dfrac{(x-y)^4 + (x+y)^7}{(x-y)^2}$, without typing $r$. /10

9. Are the expressions $p = \dfrac{x^2 - 6x + 9}{x - 3}$ and $q = x - 3$ the same?

   Justify your answer by appropriate *symbolic* computations.

   Demonstrate the application of a *numerical* probability-one equality test. /10