

Normalizing Expressions

1 Normal and Canonical Form

- deciding whether two expressions are the same
- normalizing an expression

2 Rewriting Multivariate Polynomials

- variable and term orders
- a numerical test on equality

MCS 320 Lecture 15
Introduction to Symbolic Computation
Jan Verschelde, 28 June 2024

Normalizing Expressions

1 Normal and Canonical Form

- deciding whether two expressions are the same
- normalizing an expression

2 Rewriting Multivariate Polynomials

- variable and term orders
- a numerical test on equality

Are Two Expressions The Same?

Consider

$$p = x(1 + y) \quad \text{and} \quad q = x + xy.$$

Are p and q the same?

- As expressions p and q are different.
- But mathematically, p and q are the same.

For $p = x(1 + y)$ and $q = x + xy$, the difference is

$$x(1 + y) - x - xy.$$

A similar question is:

Does an expression reduce to zero?

Normalizing Expressions

1 Normal and Canonical Form

- deciding whether two expressions are the same
- normalizing an expression

2 Rewriting Multivariate Polynomials

- variable and term orders
- a numerical test on equality

Normalizing an Expression

Definition (Normal Form)

A *normal form* of an expression gives a unique representation of the expression.

Bringing an expression into normal form is *normalizing an expression*.

Consider a polynomial p in one variable.

- The Horner form is a normal form of p .
- Another normal form of p is obtained by
 - 1 fully expanding the polynomial,
 - 2 removing terms with zero coefficient, and
 - 3 sorting the monomials from high to low degree.

For rational expressions, there are four different normal forms.

The unique normal form is called *the canonical form*.

Normalizing Expressions

1 Normal and Canonical Form

- deciding whether two expressions are the same
- normalizing an expression

2 Rewriting Multivariate Polynomials

- **variable and term orders**
- a numerical test on equality

Rewriting Multivariate Polynomials

A polynomial in x, y, z is written

- 1 after choosing a variable order, e.g.: $x > y > z > 1$, and
- 2 after choosing a monomial order, for example:
 - ▶ pure lexicographic, not considering any degrees,
 - ▶ degree lexicographic:
 - 1 order monomials first by degree,
 - 2 break ties lexicographically.

Examples:

- $x^3 + 2x^2y + 3x^2 + 4xy^2 + 5x + 6z^3$

is in pure lexicographic order.

- $x^3 + 2x^2y + 4xy^2 + 6z^3 + 3x^2 + 5x$

is in degree lexicographic order.

Normalizing Expressions

1 Normal and Canonical Form

- deciding whether two expressions are the same
- normalizing an expression

2 Rewriting Multivariate Polynomials

- variable and term orders
- a numerical test on equality

A Numerical Test on Equality

Let f and g be two polynomials in x , y , and z .

To test whether $f = g$, rewrite f and g using the *same* monomial order.

The number of monomials in several variables grows exponentially as the number of variables increase.

A numerical test proceeds as follows:

- 1 Generate a random point with coordinates (a, b, c) .
- 2 Compare $f(x = a, y = b, z = c)$ with $g(x = a, y = b, z = c)$.

This numerical test does not require rewriting and can run faster.