B-Trees

1. B-Trees
   - nodes with many children
   - inserting *into* a 2-3-4 node
   - a type node
   - a class for B-trees

2. manipulating a B-tree
   - an elaborate example
   - the insertion algorithm
   - removing elements
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balancing search trees

We start with some terminology and an overview:

- A 2-node has two children. A 3-node has three children. A 4-node has four children.
- A binary tree is also called a 2-tree as all its nodes are 2-nodes. The nodes in a 2-3 tree are 2-nodes or 3-nodes. The nodes in a 2-3-4 tree are 2-nodes, 3-nodes, or 4-nodes. Consider the red-black tree as a 2-3-4 tree in binary format.
- Balance is maintained in a 2-3 tree and 2-3-4 tree as the insert, instead of hanging a new node onto a leaf, inserts a new node into a leaf.
- A B-tree allows for up to $n$ children per node, $n > 2$. Applications for B-trees are indices to databases.

As the 2-3-4 tree is a special case of a B-tree, we focus on B-trees.
an example of storing ordered data

A tree with all its 4-nodes filled:

```
  4  8  12
     /   \
   13 14 15
     /   \
  9  10 11
     /   \
  5  6  7
     /   \
  1  2  3
```

A B-tree is a search tree:

1. data at the nodes are sorted, so we may use binary search;
2. every data element has a left and a right child;
3. at left: less than data element, at right: larger.
relation with red-black trees

The equivalent 2-3-4 tree to the red-black tree is

- A node with no red children is a 2-node.
- A node with one red child is a 3-node.
- A node with two red children is a 4-node.
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Inserting into a 2-3-4 node

Inserting 42, 99, and 80 into a 2-3-4 tree:

\[
\begin{array}{c}
42 \\
42 99 \\
42 80 99
\end{array}
\]

Inserting 64 causes the 4-node to split:

\[
\begin{array}{c}
42 80 99 \\
80 \quad 99 \\
42 \\
80 \quad 99 \\
42 64
\end{array}
\]

The split of the 4-node makes room as the leaves are 2-nodes.

Inserting 12:

\[
\begin{array}{c}
80 \\
99 \\
42 64
\end{array}
\]

\[
\begin{array}{c}
80 \\
99 \\
12 42 64
\end{array}
\]
Inserting 47 causes a split of the 4-node:

1. The middle of the splitted 4-node (that is the number 42) is inserted into the root.
2. The split of the 4-node makes room for the new number 47.
After inserting 66, we insert 52:

1. The middle of the splitted 4-node (that is the number 64) is inserted into the root.

2. The split of the 4-node makes room for the new number 52.
inserting *into* a 2-3-4 node continued

After inserting 60, the insert 50 splits the root node:

As 50 > 42, it causes a split of another 4-node.
inserting \textit{into} a 2-3-4 node continued

- We split a 4-node to make room for 50:

Observe that the tree is balanced.
Why do we have a balanced search tree?

At the insert of a new number, as we navigate the search tree:

1. As soon as we reach the first (full) 4-node we split.

   If that first 4-node is the root, then we increase the depth of the search tree by one. Because we are splitting the root, the length of any path from the root to the any leaf increases by one. If the tree was balanced, then the tree remains balanced.

   Otherwise, if the first 4-node is not the root, then there will be space to insert the item which pops out by the split, to insert that item into the parent node, because that parent node is not a full 4-node. In this case, the depth does not increase and the tree remains balanced.

2. After the splits, there is sufficient space to insert the new number at a leaf of the search tree.
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#ifndef MCS360_BTREE_NODE_H
#define MCS360_BTREE_NODE_H
#define NULL 0

struct Node
{
    Item_Type data[numbchil-1];
    // data stored at node
    int size;
    // number of data items, 0 <= size < numbchil-1

    Node* child[numbchil]; // pointers to children

    // for all i: 0 <= i < size :
    // d in subtree child[i] : d < data[i]
    // and
    // d > data[size-2], for d in child[size-1].
Node() // constructor of node with no data
{
    size = 0;
    for(int i=0; i<numbchil; i++)
    {
        if(i < numbchil-1) data[i] = Item_Type();
        child[i] = NULL;
    }
}

#endif
testing the node

#define Item_Type int
#define numbchil 5
#include "mcs360_btree_node.h"

int main()
{
    Node nd;

    for(int i=0; i<numbchil-1; i++)
        nd.data[i] = i+1;

    cout << "Data at the node :";
    for(int i=0; i<numbchil-1; i++)
        cout << " " << nd.data[i];
    cout << endl;

    return 0;
}
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a class B_Tree

namespace mcs360_btree
{
    template<typename Item_Type, int numbchil>
    class B_Tree
    {
        private:
            #include "mcs360_btree_node.h"
            Node *root;    // data member
        public:
            B_Tree();
            B_Tree(int n, Item_Type *a);
                // creates a tree with n items in a
                // precondition: n < numbchil
            B_Tree(int n, Item_Type *a,
                    B_Tree<Item_Type,numbchil> *c);
                // creates a tree with n items in a
                // and n+1 children in c
                // precondition: n < numbchil
template <typename Item_Type, int numbchil> 
B_Tree<Item_Type, numbchil>::B_Tree ( int n, Item_Type *a, 
    B_Tree<Item_Type, numbchil> *c ) 
{
    root = new Node;
    root->size = n;
    for(int i=0; i<n; i++)
    {
        root->data[i] = a[i];
        root->child[i] = c[i].root;
    }
    root->child[n] = c[n].root;
}
selectors for data

template < typename Item_Type, int numbchil >
int B_Tree<Item_Type,numbchil>::get_size()
{
    return root->size;
}

template < typename Item_Type, int numbchil >
Item_Type* B_Tree<Item_Type,numbchil>::get_data()
{
    return root->data;
}
selectors for children

template < typename Item_Type, int numbchil >
bool B_Tree<Item_Type,numbchil>::is_null_child(int k) const
{
    return (root->child[k] == NULL);
}

template < typename Item_Type, int numbchil >
B_Tree<Item_Type,numbchil>::get_child(int k) const
{
    B_Tree<Item_Type,numbchil> b;
    b.root = root->child[k];
    return b;
}
private:
    int binary_search
    (int first, int last, Item_Type i);

    // Applies binary search to find i in the array
    // starting at first and ending at last.

    // The index on return is either where i occurs,
    // or the child where i should be inserted.

public:
    int search ( Item_Type i );

    // returns the index where i occurs,
    // otherwise returns the child where i
    // could be inserted
template < typename Item_Type, int numbchil >
int B_Tree<Item_Type,numbchil>::binary_search
(int first, int last, Item_Type i)
{
    if(first == last)
        return first;
    else
    {
        int middle = (first + last)/2;

        if(i == root->data[middle])
            return middle;
        else if(i < root->data[middle])
            return binary_search(first,middle,i);
        else
            return binary_search(middle+1,last,i);
    }
}
the definition of search

template < typename Item_Type, int numbchil >
int B_Tree<Item_Type,numbchil>::search ( Item_Type i )
{
    int L = root->size-1;
    if(i > root->data[L])
        return L+1;
    else
        return this->binary_search(0,L,i);
}
writing trees

```cpp
void write_data ( B_Tree<int,5> T )
{
    int *d = T.get_data();

    for(int i=0; i<T.get_size(); i++)
    {
        cout << " " << d[i];
    }
    cout << endl;
}

void write ( int k, B_Tree<int,5> T )
{
    for(int i=0; i<k; i++) cout << " ";
    write_data(T);
    for(int i=0; i<T.get_size()+1; i++)
    {
        if(!T.is_null_child(i))
            write(k+1, T.get_child(i));
    }
}
```
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an example
nodes have 5 children

Inserting 1, 2, 3, 4:

1 2 3 4

At the start we insert at the root, and maintain the inserted elements in order.

If the node is full, we split the node at the middle.
an example
nodes have 5 children

Inserting 5 into:

```
  1 2 3 4
```

caused a split in the middle, 3 is new parent:

```
  3
  4 5
  1 2
```

Next we insert 6 and 7 ...
After inserting 6 and 7, if we insert 8 into:

```
3
```

then we cause a split, with 6 as new parent:
After inserting 9, 10, and 11:

![Diagram of a B-tree after inserting 9, 10, and 11]

Then we insert 12, 13, and 14 . . .
an example
nodes have 5 children

After inserting 12, 13, and 14:

We continue, inserting 15, 16, and 17 ...
an example

nodes have 5 children
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the insertion algorithm

A B-tree maintains its balance by
- splitting full nodes in half,
- inserting middle element into the parent.

Recursive algorithm to insert:
1. find node where to insert,
   decide if simple insert or if split is needed
2. perform simple insert or split and return results
3. after returning from insert, check if split
   place new child and insert item moved up to parent
inserting a sequence of numbers

Insert 48, 84, 43, 91, 38, 79, 63, 59, 49, 98 into a B-tree with $n = 4$.

After inserting 48:

\[
\begin{array}{c}
48 \\
\end{array}
\]

After inserting 84:

\[
\begin{array}{c}
48 \\
84
\end{array}
\]

After inserting 43:

\[
\begin{array}{c}
43 \\
48 \\
84
\end{array}
\]

The insert of 91 causes a split in the middle, 48 is the new root:

\[
\begin{array}{c}
48 \\
\end{array}
\]

\[
\begin{array}{c}
84 \\
91
\end{array}
\]

\[
\begin{array}{c}
43 \\
\end{array}
\]

Introduction to Data Structures (MCS 360)
 insert 38, 79, 63, 59, 49, 98

After inserting 38:

After inserting 79:
inserting 63 causes a split

The split leads to an insert of 84 into the root node.

After inserting 63:
inserting 59, 49

After inserting 59:

Inserting 49 will cause a split.
inserting 49 causes a split

The insert of 49 causes the insert of 63 into the root node:
finally, we insert 98

After inserting 98:
The red-black tree equivalent to the B-tree is
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removing elements

Two cases: item is at leaf or at internal node.

If item is at leaf, then

1. remove the item from the data array,
2. if leaf is less than half full, redistribute,
3. if leaf and its sibling are half full, merge leaf, sibling, and parent in one node.

If item is at internal node, then replace it by its inorder predecessor that is a leaf.
Ended chapter 11 on balancing binary search trees.

Exercises:

1. Generate a random sequence of 16 numbers and insert the numbers into a B-tree where all nodes have 5 children. Draw all intermediate steps.

2. Find the order in which the items were inserted with the first example, that is: what order of 1, 2, ..., 15 gives a tree with all its 4-nodes filled.

3. Write code to count the total number of items stored in a B-tree.