Balancing Search Trees

1. Tree Balance and Rotation
   - binary search trees
   - right rotation of a tree around a node
   - code for right rotation

2. AVL Trees
   - self-balancing search trees
   - four kinds of critically unbalanced trees

3. code for rotation
   - from left-right to left-left tree

MCS 360 Lecture 33
Introduction to Data Structures
Jan Verschelde, 13 April 2020

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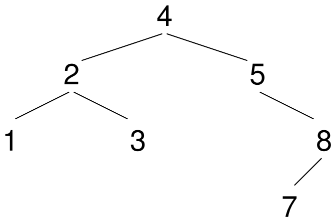
Binary Search Trees

Consider 4, 5, 2, 3, 8, 1, 7 (recall lecture 24).

Insert the numbers in a tree:



Rules to insert x at node N:

- if N is empty, then put x in N
- if $x < N$, insert x to the left of N
- if $x \geq N$, insert x to the right of N

Recursive printing: left, node, right sorts the sequence.

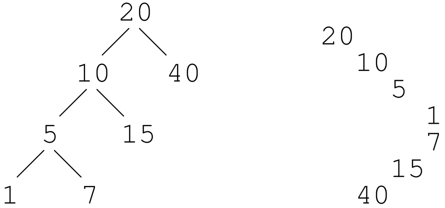
an unbalanced tree

Inserting $0, 1, 2, \ldots, 9$.

```
depth of tree : 9
0
  1
    2
      3
        4
          5
            6
              7
                8
                  9
```

shaping binary search trees

To make a binary search tree with given shape:



Insert numbers in a particular order: 20, 40, 10, 5, 15, 1, 7.

$$\begin{array}{rcl} \mathrm{depth}(T) & = & 0, \text{ if } T \text{ is empty}, \\ & = & 1 + \max(\mathrm{depth}(\mathrm{left}(T)), \mathrm{depth}(\mathrm{right}(T))), \text{ otherwise.} \end{array}$$

The tree is unbalanced because the depth of the left tree is two, while the depth of the right tree is zero.

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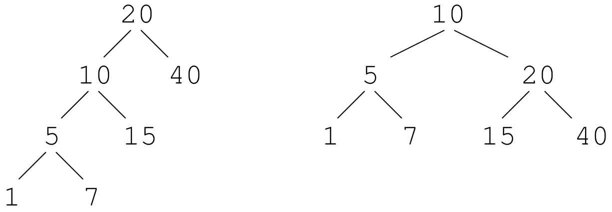
Right Rotation

To balance the binary search tree,
we do a right rotate around the root:

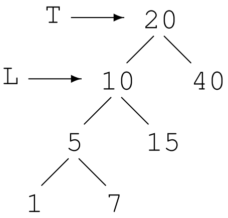


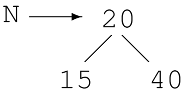
Observe the effects of a right rotation:

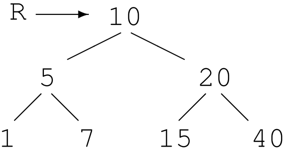
- left tree has become the new root;
- old root is now at the right of new root;
- left tree of old root is now the right tree of the left tree of old root.

Right Rotation in 3 Steps

Tree with root node `T`:







1. Label left of `T` with `L`.
2. New tree `N` has right of `T` as right and as left the right of `L`.
3. Result `R` has `L` as root, the tree `N` as right, and the left of `L` as left.

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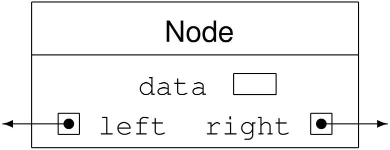
a node struct




```
struct Node
{
   int data;     // numbers stored at node in tree
   Node *left;   // pointer to left branch of tree
   Node *right;  // pointer to right branch of tree

   Node(const int& item, Node* left_ptr = NULL,
                         Node* right_ptr = NULL) :
      data(item),
      left(left_ptr), right(right_ptr) {}
```

a class Tree

```
#include "mcs360_integer_tree_node.h"

namespace mcs360_integer_tree
{
  class Tree
  {
     private:
       Node *root;  // data member

     public:
       Tree(const int& item,
            const Tree& left = Tree(),
            const Tree& right = Tree() ) :
     root(new Node(item,left.root,right.root)) {}
       Tree get_left() const;
       Tree get_right() const;
       void insert(int item);
```

function `rotate_right`

Prototype of function in client of class Tree:

```
Tree rotate_right ( Tree t );

// Returns the tree rotated to the right
// around its root.

// Precondition: left of t is not null.
```

definition of rotate_right

```
Tree rotate_right ( Tree t )
{
   Tree left = t.get_left();

   Tree new_t = Tree(t.get_data(),
      left.get_right(),t.get_right());

   Tree R = Tree(left.get_data(),
                 left.get_left(),new_t);

   return R;
}
```

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AVL Trees

Define the balance of a tree as

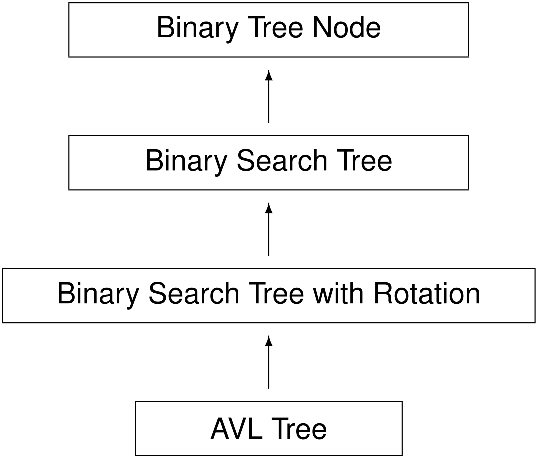
$$\text{balance} = \text{depth(right tree)} - \text{depth(left tree)}.$$

G.M. Adel'son-Vel'skiĭ and E.M Landis published in 1962 an algorithm to maintain the balance of a binary search tree.

If balance gets out of range $-1 \ldots +1$,
the subtree is rotated to bring into balance.

Their approach is known as *AVL trees*.

a Class Hierarchy

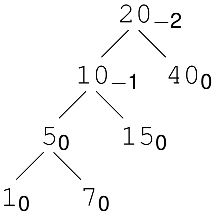


computing the balance

Recall the definition:

$$\text{balance} = \text{depth(right tree)} - \text{depth(left tree)}.$$

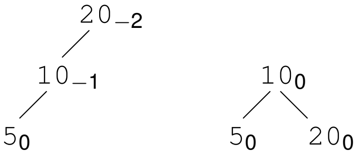
At every node we compute the balance,
displayed as subscript:



balancing a left-left tree

The tree below is *left heavy* as the balance is -2.

We also say that this is a *left-left tree*.



Executing a right rotation balances the tree.

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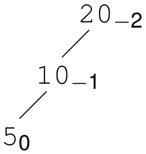
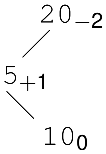
critically unbalanced trees

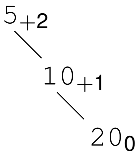
A tree is *critically unbalanced* if its balance is -2 or $+2$.



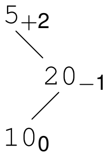
a left-left tree



a left-right tree



a right-right tree



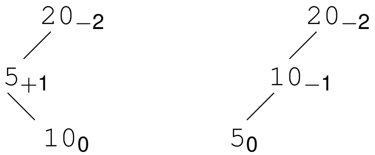
a right-left tree

balancing trees of mixed kind

A right rotation balances a left-left tree
and a left rotation balances a right-right tree.

Balancing a left-right tree happens in two stages:

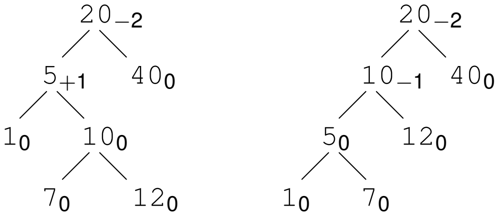
1. rotate left-right tree to left-left tree:



2. apply right rotation to left-left tree:

rotating a left-right tree

We rotate the left-right tree to a left-left tree:

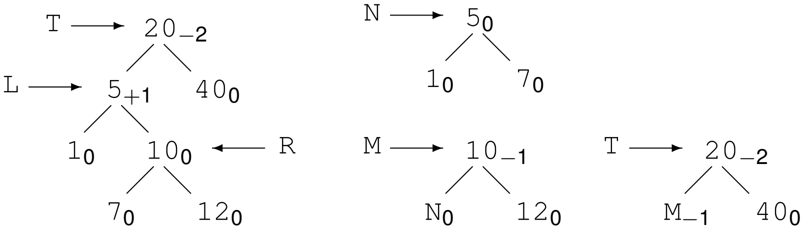


Observe the effects of the rotation:

- the data at the left node of the new tree (10)
  is swapped with the data of the left of the old tree (5);
- the right of the left of the new tree (12)
  is the right of the right of the left of the old tree;
- the right of the left of the left of the new tree (7)
  is the left of the right of the left of the old tree.

rotating to left-left tree in 4 steps

Tree with root node `T`:



1. Label left of `T` with `L` and right of `L` with `R`.
2. Tree `N` has as its left the left of `L`, as its right the left of `R`.
3. Tree `M` has as its left `N`, as its right the right of `R`.
4. Return the tree with its left `M` and its right the right of `T`.

a function to rotate a tree

```
Tree balance_by_rotation ( Tree t )
{
   if(is_left_left(t))
      return rotate_right(t);
   else if(is_right_right(t))
      return rotate_left(t);
   else if(is_left_right(t))
   {
      Tree R = rotate_to_left_left(t);
      return rotate_right(R);
   }
   else if(is_right_left(t))
   {
      Tree R = rotate_to_right_right(t);
      return rotate_left(R);
   }
```

the recursive calls

```
   else
   {
      Tree L,R;
      if(!t.is_left_null())
         L = balance_by_rotation(t.get_left());
      if(!t.is_right_null())
         R = balance_by_rotation(t.get_right());
      return Tree(t.get_data(),L,R);
   }
}
```

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prototype of the function

```
Tree rotate_to_left_left ( Tree t );

// Returns the tree rotated to a left-left tree.

// Preconditions:
//   (1) left of t is not null; and
//   (2) right of left of t is not null.
```

Test: insert 20, 5, 1, 10, 7, 12 to binary search tree.

definition of the function

```
Tree rotate_to_left_left ( Tree t )
{
   Tree left = t.get_left();
   Tree right = left.get_right();

   Tree new_left = Tree(left.get_data(),
      left.get_left(),right.get_left());

   Tree new_right = Tree(right.get_data(),
      new_left,right.get_right());

   Tree R = Tree(t.get_data(),
      new_right,t.get_right());

   return R;
}
```

rebalancing search trees

After each insert (or removal):

- check the balance of the tree,
- and if critically unbalanced, rebalance.

Performance of the AVL tree:

- worst case: $1.44 \times \log_2(n)$,
- on average: $\log_2(n) + 0.25$ comparisons needed.

$\rightarrow$ close to complete binary search tree.

Summary + Exercises

Started chapter 11 on balancing binary search trees.

Exercises:

1. Take a numerical example to left rotate a binary search tree with integer values. Formulate carefully each step in the left rotation. Justify the correctness of the algorithm.
2. Formulate the algorithm to rotate a right-left tree to a right-right tree and illustrate with an example.
3. The posted code provides functions to make an AVL tree. Design a class to represent an AVL tree.
4. Take your design of the previous exercise and define the methods of the class to represent an AVL tree.