Recursive Definitions

1. Recursive Mathematical Formulas
   - the factorial of a natural number
   - tracing a recursive execution
   - an accumulating parameter

2. Recursion on STL Lists
   - generating n random numbers
   - writing a list recursively
   - searching a list

3. Recursive Greatest Common Divisor
   - computing the greatest common divisor recursively

MCS 360 Lecture 21
Introduction to Data Structures
Jan Verschelde, 2 March 2020
Recursive Definitions

1. Recursive Mathematical Formulas
   - the factorial of a natural number
   - tracing a recursive execution
   - an accumulating parameter

2. Recursion on STL Lists
   - generating n random numbers
   - writing a list recursively
   - searching a list

3. Recursive Greatest Common Divisor
   - computing the greatest common divisor recursively
Given a natural number $n$, its factorial $n!$ is

$$n! = n \times (n - 1) \times \cdots 2 \times 1.$$ 

Interpretation: #choices of $n$ items without repetition. For example: how many 3-letter words with a, b, c?

abc, acb, bac, bca, cab, cba # : $3! = 6$.

What is $0!$? How many ways to choose nothing? $0! = 1$.

A recursive formula for $n!$ is

$$n! = \begin{cases} 
1 & \text{if } n = 0, \\
n \times (n - 1)! & \text{if } n > 0.
\end{cases}$$
int factorial(int n);
// returns the factorial of n

int factorial(int n)
{
    if(n==0)
        return 1;
    else
        return n*factorial(n-1);
}
Computing 5! recursively happens via a stack

- not in the base case: push argument on the stack
- after base case, pop from stack and evaluate
unwinding the recursion

factorial(5)
→ factorial(4)
  → factorial(3)
    → factorial(2)
      → factorial(1)
        → factorial(0)
          return 1
          return 1 * 1
          return 2 * 1 * 1
          return 3 * 2 * 1 * 1
        return 4 * 3 * 2 * 1 * 1
      return 5 * 4 * 3 * 2 * 1 * 1
Recursive Definitions

1. Recursive Mathematical Formulas
   - the factorial of a natural number
   - tracing a recursive execution
   - an accumulating parameter

2. Recursion on STL Lists
   - generating n random numbers
   - writing a list recursively
   - searching a list

3. Recursive Greatest Common Divisor
   - computing the greatest common divisor recursively
tracing recursion

tracing 5! ...

    n = 5
    n = 4
    n = 3
    n = 2
    n = 1
n = 0
returning 1
  returning 2
    returning 6
      returning 24
        returning 120
the function `trace_factorial`

```cpp
int trace_factorial(int n)
{
    for(int i=0; i<n; i++) cout << " ";
cout << "n = " << n << endl;

    if(n==0)
    {
        return 1;
        cout << "returning 1" << endl;
    }
    else
    {
        int r = n*trace_factorial(n-1);
        for(int i=0; i<n; i++) cout << " ";
cout << "returning " << r << endl;
        return r;
    }
}
```
Recursive Definitions

1. Recursive Mathematical Formulas
   - the factorial of a natural number
   - tracing a recursive execution
   - an accumulating parameter

2. Recursion on STL Lists
   - generating n random numbers
   - writing a list recursively
   - searching a list

3. Recursive Greatest Common Divisor
   - computing the greatest common divisor recursively
Instead of a tail recursion for \( n! \), we can accumulate the result in a parameter.

```c
int accumulate_factorial(int n, int f)
{
    if(n <= 1)
        return f;
    else
        return accumulate_factorial(n-1,n*f);
}
```
tracing the execution

computing 5*1
computing 4*5
computing 3*20
computing 2*60
returning 120

```c
int trace_accumulate_factorial(int n, int f)
{
    if(n <= 1)
    {
        cout << "returning " << f << endl;
        return f;
    }
    else
    {
        cout << "computing " << n << "*" << f << endl;
        return trace_accumulate_factorial(n-1,n*f);
    }
}
```
Recursive Definitions

1. Recursive Mathematical Formulas
   - the factorial of a natural number
   - tracing a recursive execution
   - an accumulating parameter

2. Recursion on STL Lists
   - generating n random numbers
   - writing a list recursively
   - searching a list

3. Recursive Greatest Common Divisor
   - computing the greatest common divisor recursively
generating \( n \) random numbers

A recursive view of a nonempty list \( \mathbb{L} \):
\( \mathbb{L} \) has a node as head and a list as tail.

A recursive algorithm to generate \( n \) numbers:

- If \( n \) equals zero (or less) then
  
  return an empty list;

- else (\( n \) is larger than zero)
  
  ▶ generate a list \( \mathbb{L} \) of \( n-1 \) numbers;
  
  ▶ push a random number to \( \mathbb{L} \).
the function \texttt{generate}

\begin{verbatim}
list<int> generate ( int n )
{
    if(n <= 0)
    {
        list<int> L;
        return L;
    }
    else
    {
        int r = rand() \% 1000;
        list<int> L = generate(n-1);
        L.push_front(r);
        return L;
    }
}
\end{verbatim}
tracing the execution

Making `generate` verbose with print statements:

generating 3 random numbers ...
generate with n = 3 ...
generate with n = 2 ...
generate with n = 1 ...
generate with n = 0 ...
returning empty list
pushing 73 to front ...
pushing 249 to front ...
pushing 807 to front ...

What is the content of the list on return?
Recursive Definitions

1. Recursive Mathematical Formulas
   - the factorial of a natural number
   - tracing a recursive execution
   - an accumulating parameter

2. Recursion on STL Lists
   - generating n random numbers
   - writing a list recursively
   - searching a list

3. Recursive Greatest Common Divisor
   - computing the greatest common divisor recursively
writing a list recursively

A recursive algorithm to write a list $L$:

- if list $L$ is empty then
  we do nothing;

- else ( $L$ is not empty )
  - pop first item $i$ from $L$;
  - write $i$;
  - write $L$; // we have popped $i$
the function `write`

```cpp
void write ( list<int> L )
{
    if(!L.empty())
    {
        list<int> K = L;
        cout << " " << K.front();
        K.pop_front();
        write(K);
    }
}
```
Recursive Definitions

1. Recursive Mathematical Formulas
   - the factorial of a natural number
   - tracing a recursive execution
   - an accumulating parameter

2. Recursion on STL Lists
   - generating n random numbers
   - writing a list recursively
   - searching a list

3. Recursive Greatest Common Divisor
   - computing the greatest common divisor recursively
searching a list recursively

Does a number \( e \) belong to a list \( L \)?

- if the list \( L \) is empty then
  return false;
- else if front of \( L \) equals \( e \) then
  return true;
- else
  ▶ pop front element from list \( L \);
  ▶ return belongs \( e \) to \( L \)? // \( L \) is smaller
the function `belongs`

```cpp
bool belongs ( list<int> L, int e )
{
    if(L.empty())
        return false;
    else if(L.front() == e)
        return true;
    else
    {
        list<int> K = L;
        K.pop_front();
        return belongs(K,e);
    }
}
```
Recursive Definitions

1. Recursive Mathematical Formulas
   - the factorial of a natural number
   - tracing a recursive execution
   - an accumulating parameter

2. Recursion on STL Lists
   - generating n random numbers
   - writing a list recursively
   - searching a list

3. Recursive Greatest Common Divisor
   - computing the greatest common divisor recursively
Observe: $gcd(20,15) = gcd(15,5)$, $5 = 20 \% 15$.

A recursive algorithm to compute $gcd(a,b)$:

- base case: if $a \% b == 0$
  then $b$ divides $a$, so $b == gcd(a,b)$
- else, let $r = a \% b$, return $gcd(b,r)$.

Why is this an algorithm?
- termination: $r < b$; if $b > a$, then $r = a$
- correctness: $gcd(a,b) == gcd(b,a \% b)$
the function \texttt{gcd}

```c
int gcd(int a, int b)
{
    int r = a \% b;
    if(r == 0)
        return b;
    else
        return gcd(b, r);
}
```
tracing the execution

give x : 98212
give y : 44632
gcd(98212, 44632) = 4
tracing gcd(98212, 44632) :
a = 98212, b = 44632, r = 8948
a = 44632, b = 8948, r = 8840
a = 8948, b = 8840, r = 108
a = 8840, b = 108, r = 92
a = 108, b = 92, r = 16
a = 92, b = 16, r = 12
a = 16, b = 12, r = 4
a = 12, b = 4, r = 0
Summary + Exercises

Started Chapter 7 on recursive algorithms.

Exercise 1:

1. Write the definition of a function to copy the elements on a stack *recursively* to a list.

Use the prototype below.

```cpp
list<int> copy ( stack<int> stk );
// Returns a list with the same elements on the stack stk.
// The elements in the returned list occur in the same order as on the stack, the top of the stack occurs first; the bottom of the stack is the last element in the list.
```
more exercises

Additional exercises:

2. Draw the evolution of the stack of function calls for 5! computed recursively with an accumulating parameter.

3. Write a recursive function to sum a STL list of integer numbers. Give code to show that your function works.

4. Test what happens when the arguments for the gcd function would be negative numbers.