MCS 471: Formula Sheet for Final Exam

1. Secant: $x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} f(x_k)$, $k = 1, 2, \ldots$

   Convergence of Secant: $E_{k+1} \approx C E_k^{(1+\sqrt{5})/2}$, for some constant C and $E_k = x_k - x_\infty$.

2. Newton: $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$, $k = 0, 1, \ldots$

   Convergence to regular root: $E_{k+1} \approx C E_k^2$, for some constant C and $E_k = x_k - x_\infty$.

   Convergence to root of multiplicity m: $E_{k+1} \approx \frac{m-1}{m} E_k$, $E_k = x_k - x_\infty$.

   modified Newton $x_{k+1} = x_k - m \frac{f(x_k)}{f'(x_k)}$, $k = 0, 1, \ldots$, for $f(x) = (x-r)^m h(x)$, $h(r) \neq 0$.

3. Aitken: $x_k - \frac{(x_{k+1} - x_k)^2}{x_{k+2} - 2x_{k+1} + x_k}$, $k = 0, 1, \ldots$

4. Golden Section: $x_1 = ca + (1-c)b$, $x_2 = (1-c)a + cb$, with $c = \frac{-1+\sqrt{5}}{2} \approx 0.6180$.

5. norms for $\mathbf{x} \in \mathbb{R}^n$:

$$||\mathbf{x}||_1 = \sum_{i=1}^n |x_i| \qquad ||\mathbf{x}||_\infty = \max_{i=1}^n |x_i| \qquad ||\mathbf{x}||_2 = \left(\sum_{i=1}^n x_i^2 \right)^{1/2}$$

6. norms for $A \in \mathbb{R}^{n \times m}$:

$$||A||_1 = \max_{j=1}^m \sum_{i=1}^n |a_{ij}| \qquad ||A||_\infty = \max_{i=1}^n \sum_{j=1}^m |a_{ij}|$$

$$||A||_f = \left(\sum_{i=1}^n \sum_{j=1}^m a_{ij}^2 \right)^{1/2} \qquad ||A|| = \max_{||\mathbf{x}||=1} ||A\mathbf{x}||$$

7. condition number $\mathrm{cond}(A) = ||A|| ||A^{-1}||$ for $A\mathbf{x} = \mathbf{b}$, $\mathbf{r} = \mathbf{b} - A\bar{\mathbf{x}}$:

$$\frac{||\mathbf{r}||}{||\mathbf{b}||} \frac{1}{||A|| ||A^{-1}||} \leq \frac{||\mathbf{x} - \bar{\mathbf{x}}||}{||\mathbf{x}||} \leq ||A|| ||A^{-1}|| \frac{||\mathbf{r}||}{||\mathbf{b}||}$$

$$\frac{||\mathbf{x} - \bar{\mathbf{x}}||}{||\mathbf{x}||} \leq ||A|| ||A^{-1}|| \frac{||A - \bar{A}||}{||A||}$$

8. Newton for $f_1(\mathbf{x}) = 0$, $f_2(\mathbf{x}) = 0$, $\ldots$, $f_n(\mathbf{x}) = 0$:

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \Delta\mathbf{x} = - \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \vdots \\ f_n(\mathbf{x}) \end{bmatrix}, \qquad \mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \Delta\mathbf{x}, \quad k = 0, 1, \ldots$$

9. Lagrange interpolation: $l_i(x) = \prod_{\substack{j=0 \\ \neq i}}^n \frac{x - x_j}{x_i - x_j}$, $\quad p(x) = \sum_{i=0}^n l_i(x) f_i$.

10. Neville interpolation: $p_{i\dots j} = \dfrac{(x^* - x_j)p_{i\dots j-1} - (x^* - x_i)p_{i+1\dots j}}{x_i - x_j}$.

11. Divided differences: $f_{0\dots ji} = \frac{f_{0\dots j-1j} - f_{0\dots j-1i}}{x_j - x_i}$
$p(x) = f_0 + f_{01}(x - x_0) + f_{012}(x - x_0)(x - x_1) + \cdots + f_{012\dots n}(x - x_0)(x - x_1)\cdots(x - x_{n-1})$.

12. Interpolation error: $E(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n)$.

13. Chebyshev polynomials: $T_n(x) = \cos(n \arccos(x))$
$T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n > 0$.

14. Interpolating splines: $h_i S_{i+1} + 2(h_{i-1} + h_i)S_i + h_{i-1}S_{i-1} = 6(f[x_i, x_{i+1}] - f[x_{i-1}, x_i])$, $x_i = x_{i-1} + h_i$,
$g(x) = g_i(x), x \in [x_i, x_{i+1}]$, $g_i(x_i) = f_i$, $i = 0, 1, \dots, n-1$, $g_{n-1}(x_n) = f_n$,
$g_i(x) = a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i$,
$S_i = g_i''(x_i)$, $a_i = \frac{S_{i+1} - S_i}{6h_i}$, $b_i = S_i/2$, $c_i = \frac{f_{i+1} - f_i}{h_i} - (2S_i + S_{i+1})\frac{h_i}{6}$, $d_i = f_i$.

15. Taylor: $f(x + h) = f(x) + hf'(x) + h^2\frac{f''(x)}{2!} + h^3\frac{f^{(3)}(x)}{3!} + O(h^4)$.
Maclaurin: $f(0 + h) = f(0) + hf'(0) + h^2\frac{f''(0)}{2!} + h^3\frac{f^{(3)}(0)}{3!} + O(h^4)$.

16. $If(x) = f(x), \quad Df(x) = \frac{\partial f}{\partial x}, D = \frac{\partial}{\partial x}, \quad Ef(x) = f(x + h), E^{-1}f(x) = f(x - h)$.
$\Delta f(x) = f(x + h) - f(x)$, $\Delta = E - I$. $\nabla f(x) = f(x) - f(x - h)$, $\nabla = I - E^{-1}$.
$\delta f(x) = f(x + h) - f(x - h)$, $\delta = E - E^{-1}$.

17. $D = \frac{1}{h}\left(\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \frac{\Delta^5}{5} - \cdots\right) \quad D = \frac{1}{h}\left(\nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} + \frac{\nabla^4}{4} + \frac{\nabla^5}{5} + \cdots\right)$

18. Richardson extrapolation ($0 < r < 1$):
$\Delta f(x, h) = \frac{1}{h}\Delta f(x) \quad \Delta f(x, h, rh, \dots, r^n h) = \frac{\Delta f(x,h,rh,\dots,r^{n-1}h)r^n - \Delta f(x,rh,r^2h,\dots,r^nh)}{r^n - 1}$
$\delta f(x, h) = \frac{1}{2h}\delta f(x) \quad \delta f(x, h, rh, \dots, r^n h) = \frac{\delta f(x,h,rh,\dots,r^{n-1}h)r^{2n} - \delta f(x,rh,r^2h,\dots,r^nh)}{r^{2n} - 1}$

19. Trapezoidal rule: $\displaystyle\int_a^b f(x)dx = \frac{f(a) + f(b)}{2}(b - a)$,
composite Trapezoidal rule: $\displaystyle T(h) = \frac{h}{2}(f(a) + f(b)) + h\sum_{k=1}^{n-1} f(a + kh), \quad h = \frac{b-a}{n}$.
Romberg integration: $T[i][j] = \frac{T[i][j-1]2^{2j} - T[i-1][j-1]}{2^{2j} - 1}, \quad T[i][0] = T\left(\frac{h}{2^i}\right)$.

20. Euler-Maclaurin summation formula, for $g \in C^{2m+2}[0, N]$:

$$\frac{1}{2}g(0) + g(1) + \cdots + g(N-1) + \frac{1}{2}g(N) = \int_0^N g(t)dt$$
$$+ \sum_{l=1}^{m} \frac{B_{2l}}{(2l)!}\left(g^{(2l-1)}(N) - g^{(2l-1)}(0)\right) + \frac{B_{2m+2}}{(2m+2)!}Ng^{(2m+2)}(\alpha), \quad \alpha \in [0, N],$$

where B_k are the Bernoulli numbers, defined as $B_k = B_k(0)$. The Bernoulli polynomials $B_k(t)$ satisfy $B_{k+1}'(t) = (k + 1)B_k(t)$, with $B_{2k+1}(0) = 0$, $B_{2k+1}(1) = 0$, for all $k > 0$.

21. Fourier series: $F(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(\pi k t) + b_k \sin(\pi k t) = \sum_{k=-\infty}^{\infty} c_k e^{i\pi k t}$.
$a_k = \int_{-1}^{+1} f(t)\cos(2\pi k t)dt, k > 0, \quad a_0 = \int_{-1}^{+1} f(t)dt, \quad b_k = \int_{-1}^{+1} f(t)\sin(\pi k t)dt$
$c_k = \frac{1}{2}(a_k - ib_k), \quad c_{-k} = \frac{1}{2}(a_k + ib_k)$.

22. Euler's method: $y_{n+1} = y_n + hf(x_n, y_n)$ to solve $\frac{dy}{dx} = f(x, y(x))$
and the modified Euler's method: $y_{n+1} = y_n + \frac{h}{2}\left(f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right)$.

23. A third-order Runge-Kutta formula to solve $\frac{dy}{dx} = f(x, y(x))$:

$$\begin{aligned} k_1 &= hf(x_n, y_n) \\ k_2 &= hf(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_1) \\ k_3 &= hf(x_n + \tfrac{3}{2}h, y_n + \tfrac{3}{4}k_2) \\ y_{n+1} &= y_n + \tfrac{1}{9}(2k_1 + 3k_2 + 4k_3) \end{aligned}$$

24. A fourth-order Runge-Kutta formula to solve $\frac{dy}{dx} = f(x, y(x))$:

$$\begin{aligned} k_1 &= hf(x_n, y_n) \\ k_2 &= hf(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_1) \\ k_3 &= hf(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_2) \\ k_4 &= hf(x_n + h, y_n + k_3) \\ y_{n+1} &= y_n + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \end{aligned}$$

25. Some Adams-Bashforth formulas to solve $\frac{dy}{dx} = f(x, y(x))$:

$$\begin{aligned} y_{n+1} &= y_n + \tfrac{1}{2}h(-f_{n-1} + 3f_n) \\ y_{n+1} &= y_n + \tfrac{1}{12}h(5f_{n-2} - 16f_{n-1} + 23f_n) \\ y_{n+1} &= y_n + \tfrac{1}{24}h(9f_{n-3} + 37f_{n-2} - 59f_{n-1} + 55f_n) \\ y_{n+1} &= y_n + \tfrac{1}{720}h(251f_{n-4} - 1274f_{n-3} + 2616f_{n-2} - 2774f_{n-1} + 1901f_n) \\ y_{n+1} &= y_n + \tfrac{1}{1440}h(-475f_{n-5} + 2877f_{n-4} - 7298f_{n-3} + 9982f_{n-2} - 7923f_{n-1} + 4277f_n) \end{aligned}$$

26. Some Adams-Moulton formulas to solve $\frac{dy}{dx} = f(x, y(x))$:

$$\begin{aligned} y_{n+1} &= y_n + \tfrac{1}{2}h(f_n + f_{n+1}) \\ y_{n+1} &= y_n + \tfrac{1}{12}h(-f_{n-1} + 8f_n + 5f_{n+1}) \\ y_{n+1} &= y_n + \tfrac{1}{24}h(f_{n-2} - 5f_{n-1} + 19f_n + 9f_{n+1}) \\ y_{n+1} &= y_n + \tfrac{1}{720}h(-19f_{n-3}106f_{n-2} - 264f_{n-1} + 646f_n + 251f_{n+1}) \\ y_{n+1} &= y_n + \tfrac{1}{1440}h(27f_{n-4} - 173f_{n-3}482f_{n-2} - 798f_{n-1} + 1427f_n + 475f_{n+1}) \end{aligned}$$

27. A central-difference approximation: $f''(x_i) = \frac{f(x_i+h) - 2f(x_i) + f(x_i-h)}{h^2} + O(h^2)$, $h > 0$.