1. Consider a floating-point number system with base 10, 3 digits in the fraction (mantissa) and exponents ranging between -11 and $+11$.

   (a) Represent the numbers 56789 and 734.21 in this number system.
   Show how to calculate their sum.

   (b) For which type of operands leads the addition to the greatest loss in accuracy? /25

2. Consider the points $(0.2, 0.809)$, $(0.4, 3.090)$, $(0.6, 1.978)$, and $(0.8, 2.213)$.

   (a) Set up the table of divided differences for use in Newton interpolation.

   (b) Give the Newton form of the polynomial interpolating through these points. /15

3. To find the minimum of a function f over the interval $[a, b]$, the golden section search method compares the values of f at two points x_1 and x_2 inside the interval $[a, b]$.

   (a) Depending on the outcome of the comparison, what are the two cases of the remaining interval for the next step?

   (b) Show that in both cases, the length of the interval containing the minimum is reduced by the constant factor c (see formula sheet). /10

4. Consider the fixed-point iteration $x_{k+1} = g(x_k)$, for $g(x) = \frac{1}{3}x^2 - \frac{2}{3}x + \frac{4}{3}$.

   (a) $x = g(x)$ has two fixed points: 1 and 4. What is the rate of convergence (or divergence) for x_k sufficiently close to these two fixed points?

   (b) For $x_0 = 3$, indicate on the picture below how to compute x_1, x_2 and x_3.

/15

5. What is the relation between $f''(x_0)$ and the divided difference $f[x_0, x_1, x_2]$? Justify. /10

6. Consider the matrix $A = \begin{bmatrix} 7.284\text{E}-11 & 6.924\text{E}-01 & 2.264\text{E}-01 \\ 8.548\text{E}-01 & 4.325\text{E}-01 & 2.889\text{E}-01 \\ 9.244\text{E}-01 & 6.177\text{E}-01 & 3.591\text{E}-01 \end{bmatrix}$.

   (a) Compute the LU decomposition of A with partial pivoting.

   Calculate with four decimal places, using rounding: write the answer of every step rounded to four decimal places, and use the rounded number in the calculations of the next step.

   (b) Explain why we must do pivoting. Illustrate your argument referring to the calculations above. What would have happened without pivoting?

   /25

7. Suppose the condition number of the matrix A is 10^{10} and the working precision is 14 decimal places. How many decimal places in a numerical solution of the system $A\mathbf{x} = \mathbf{b}$ can you trust? Justify your answer.

   /10

8. (a) Convert $\frac{-\frac{1}{15}x^3+x}{1-\frac{2}{5}x^2}$ into a continued-fraction expression.

   (b) Count the number of arithmetical operations to evaluate the expression in its original and its continued-fraction form.

   /15

9. (a) Approximate $\frac{d}{dx}\ln(x)$ at $x = 1$ using central differences, for $h = 0.1, 0.01, 0.001$. Work with as many decimal places as your calculator shows.

   (b) Apply Richardson extrapolation to improve the approximation.

   (c) How many decimal places in your answer are correct? Justify your estimate.

   /25

10. The formula $y_{n+1} = y_n + \frac{1}{12}h(-f_{n-1} + 8f_n + 5f_{n+1})$ is used to solve $\frac{dy}{dx} = f(x, y(x))$. Use the method of undetermined coefficients to derive this formula.

    /10

11. Consider the boundary-value problem $y''(t) - t^2y(t) = \cos(t)$, $y(0) = 1$ and $y(2) = 2$.

    Our first guess for $y'(0) = 1$ yields 9.24924 at $t = 2$. Our second guess for $y'(0) = -1$ yields 1.26074 at $t = 2$. What is your next guess for $y'(0)$ in the shooting method?

    /15

12. Suppose we want to solve an initial-value problem $y' = f(x, y)$, $y(0) = y_0$, with a predictor-corrector method using four points in each step. Calculate how many times we evaluate f to approximate $y(1.4)$, using step size $h = 0.2$.

    /10

13. Consider the characteristic-value problem, for some parameter β:

$$y''(x) - xy'(x) + \beta x^2 y(x) = 0, \quad y(0) = 0, y(1) = 0.$$

    (a) Use finite differences with $h = 0.2$ to set up the corresponding eigenvalue problem.

    (b) Explain how to find the solution with the largest eigenvalue.

    /15