Higher Order Equations

1. Higher Order Equations
    - first order differential systems
    - Euler's method on a system
    - the pendulum
2. Linear Systems
    - solution trajectories
    - spectral decomposition
3. Computer Simulations
    - a predator-prey model
    - application of the modified Euler method

MCS 471 Lecture 29
Numerical Analysis
Jan Verschelde, 28 October 2022

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higher order differential equations

The format for an n-th order differential equation is

$$y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y^{(1)} + a_0 y^{(0)} = f(x,y),$$

where

- $y^{(0)} = y = y(x)$,
- $y^{(i)} = \dfrac{d^i y}{dx^i}$, and
- $a_{n-1}, \ldots, a_1, a_0$ are constants, independent of x and y,
- f is a function of x and y.

an example of a second order equation

Consider:

$$y^{(2)} + y^{(0)} = 0,$$

where $y^{(0)} = y = y(x)$ and $y^{(2)} = \dfrac{d^2y}{dx^2}$.

A solution is $y(x) = c_1 \cos(x) + c_2 \sin(x)$, for constants c_1, c_2.

Let us verify:

$$\begin{array}{rcl} y^{(0)} & = & c_1 \cos(x) + c_2 \sin(x), \\ y^{(1)} & = & -c_1 \sin(x) + c_2 \cos(x), \\ y^{(2)} & = & -c_1 \cos(x) - c_2 \sin(x). \end{array}$$

Indeed, $y^{(2)} + y^{(0)} = 0$.

a general third order differential equation

Consider a differential equation of order three:

$$y^{(3)} + a_2 y^{(2)} + a_1 y^{(1)} + a_0 y^{(0)} = f(x, y).$$

We simplify the notation:

$$y^{(3)} + a_2 y_3 + a_1 y_2 + a_0 y_1 = f(x, y),$$

with auxiliary variables $y_1 = y^{(0)}$, $y_2 = y^{(1)}$, and $y_3 = y^{(2)}$.

Observe that $y_3 = y^{(2)}$ implies $y_3' = y^{(3)}$.

we obtain a first order differential equation:

$$\begin{array}{rcl} y_3' + a_2 y_3 + a_1 y_2 + a_0 y_1 & = & f(x, y), \\ y_3' & = & f(x, y) - a_2 y_3 - a_1 y_2 - a_0 y_1 . \end{array}$$

from third order to first order

One *third* order differential equation

$$y^{(3)} + a_2 y^{(2)} + a_1 y^{(1)} + a_0 y^{(0)} = f(x,y)$$

becomes a system of *three* *first* order differential equations:

$$\left\{ \begin{array}{rcl} y_3' & = & f(x,y) - a_2 y_3 - a_1 y_2 - a_0 y_1 \\ y_2' & = & y_3, \quad \text{as} \quad y_3 = \dfrac{d^2 y}{dx^2} \\ y_1' & = & y_2, \quad \text{as} \quad y_2 = \dfrac{dy}{dx} \end{array} \right.$$

rewrite the system in matrix-vector notation

$y^{(3)} + a_2 y^{(2)} + a_1 y^{(1)} + a_0 y^{(0)} = f$ is equivalent to

$$\left\{ \begin{array}{rcl} y_3' & = & f - a_2 y_3 - a_1 y_2 - a_0 y_1 \\ y_2' & = & y_3 \\ y_1' & = & y_2 \end{array} \right.$$

We align the variables explicitly:

$$\left\{ \begin{array}{rcrcrcrcrcr} y_1' & = & & & & & y_2 & & & & \\ y_2' & = & & & & & & & y_3 & & \\ y_3' & = & - & a_0 y_1 & - & a_1 y_2 & & - & a_2 y_3 & + & f \end{array} \right.$$

In matrix-vector notation:

$$\left[\begin{array}{c} y_1' \\ y_2' \\ y_3' \end{array} \right] = \left[\begin{array}{rrr} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{array} \right] \left[\begin{array}{c} y_1 \\ y_2 \\ y_3 \end{array} \right] + \left[\begin{array}{c} 0 \\ 0 \\ f \end{array} \right]$$

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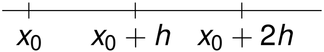
Euler's method

To solve $y' = f(x, y(x))$, $y(x_0) = y_0$, we discretize, for $h > 0$:



$$\left\{ \begin{array}{rcl} x_n & = & x_0 + nh, \\ y_n & \approx & y(x_n). \end{array} \right.$$

Euler's method: $y_{n+1} = y_n + hf(x_n, y_n)$, $n = 0, 1, \ldots$

In three dimensions, $\mathbf{f} = (f_1, f_2, f_3)$, $\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}(x, \mathbf{y}_n)$:

$$\left[\begin{array}{c} y_{n+1,1} \\ y_{n+1,2} \\ y_{n+1,3} \end{array} \right] = \left[\begin{array}{c} y_{n,1} \\ y_{n,2} \\ y_{n,3} \end{array} \right] + h \left[\begin{array}{c} f_1(x_n, y_{n,1}, y_{n,2}, y_{n,3}) \\ f_2(x_n, y_{n,1}, y_{n,2}, y_{n,3}) \\ f_3(x_n, y_{n,1}, y_{n,2}, y_{n,3}) \end{array} \right], \quad n = 0, 1, \ldots$$

starting at x_0, $y_{0,1} = y_1(x_0)$, $y_{0,2} = y_2(x_0)$, $y_{0,3} = y_3(x_0)$.

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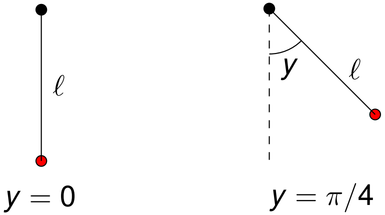
the pendulum

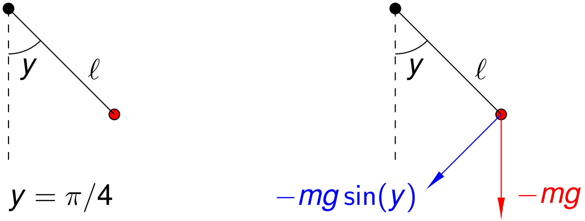
Consider a pendulum swinging under gravity.

- The length of the pendulum is a constant ℓ.
- m is the mass of the pendulum.
- The gravitational constant is g.

The variable $y(t)$ is the angle the pendulum makes with respect to its resting position $y = 0$. The initial velocity is $y'(0)$.



an initial value problem



Newton's second law of motion $F = ma$ leads to

$$m\ell\frac{d^2y}{dt^2} = -mg\sin(y).$$

For the initial value problem, we give an initial angle and initial velocity.

a system of two first order equations

The mass m does not matter, we divide by $\ell > 0$:

$$m\ell\frac{d^2y}{dt^2} = -mg\sin(y) \quad \Rightarrow \quad \frac{d^2y}{dt^2} = -\frac{g}{\ell}\sin(y).$$

Set $y_1 = y$ and let $y_2 = \dfrac{dy}{dt} = y' = y_1'$, so the first equation is

$$y_1' = y_2.$$

Use $y_2' = \dfrac{d^2y}{dt^2}$ for the second equation:

$$y_2' = -\frac{g}{\ell}\sin(y) = -\frac{g}{\ell}\sin(y_1).$$

applying Euler's method

Apply Euler's method to

$$\begin{array}{rcl} y_1' & = & y_2, \\ y_2' & = & -\dfrac{g}{\ell} \sin(y_1). \end{array}$$

For $h > 0$, $t_n = 0 + n h$, we compute

$$\begin{array}{rcl} y_{n+1,1} & = & y_{n,1} + h \, y_{n,2} \\ y_{n+1,2} & = & y_{n,2} + h \left(-\dfrac{g}{\ell} \sin(y_{n,1}) \right), \quad n = 0, 1, \ldots \end{array}$$

starting at $y_1 = \pi/4$ (initial angle), $y_2 = 0$ (initial velocity).

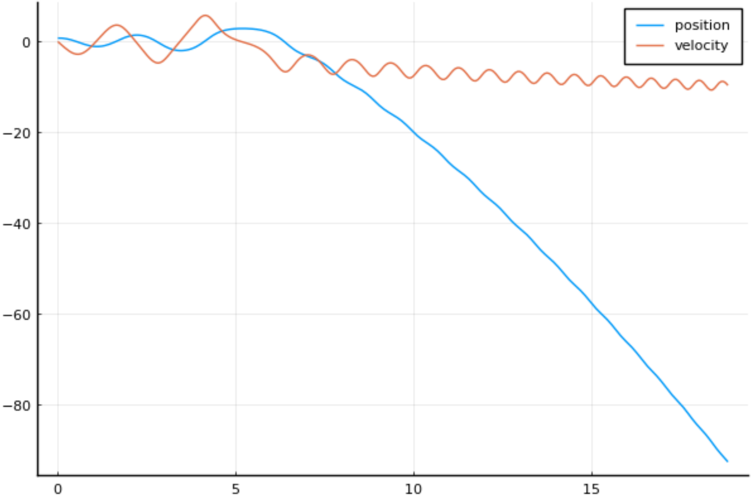
In the numerical experiments, we use $\ell = 1$ and $g = 9.807$.

a Julia function

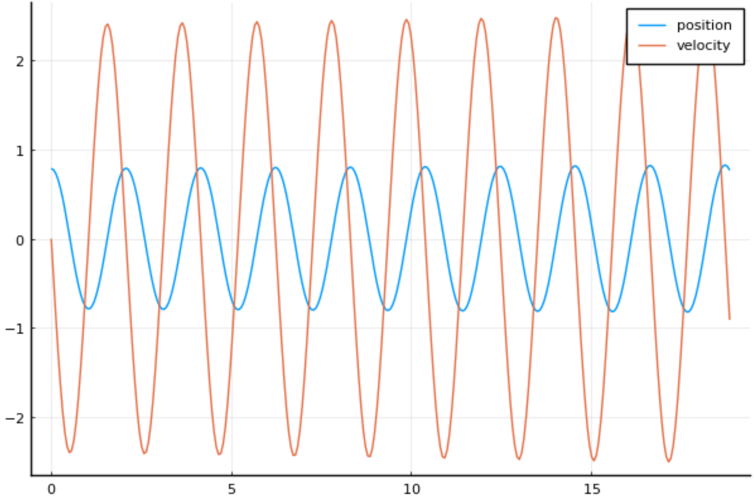
```
"""
    eulerpend(p:Int,n::Int)

applies Euler’s method with n*p evaluations
on the interval [0,p*2*pi] on the pendulum problem.
"""
function eulerpend(p::Int,n::Int)
    h = 2*p*pi/n
    (rt, ry1, ry2) = (zeros(n+1), zeros(n+1), zeros(n+1))
    y1 = pi/4
    y2 = 0
    (rt[1], ry1[1], ry2[1]) = (0, y1, y2)
    for i=1:n
        t = i*h
        (y1, y2) = (y1 + h*y2, y2 + h*(-9.807*sin(y1)))
        (rt[i+1], ry1[i+1], ry2[i+1]) = (t, y1, y2)
    end
    return (rt, ry1, ry2)
end
```

plotting angles and velocities for $(p, n) = (3, 100)$



the modified Euler method for $(p, n) = (3, 100)$



applying the modified Euler method

Apply the modified Euler's method to

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= -\frac{g}{\ell}\sin(y_1). \end{aligned}$$

For $h > 0$, $t_n = 0 + n\,h$, we compute

$$\begin{aligned} \overline{y}_{n+1,1} &= y_{n,1} + h\,y_{n,2} \\ \overline{y}_{n+1,2} &= y_{n,2} + h\left(-\frac{g}{\ell}\sin(y_{n,1})\right) \\ y_{n+1,1} &= y_{n,1} + \frac{h}{2}\left(y_{n,2} + \overline{y}_{n+1,2}\right) \\ y_{n+1,2} &= y_{n,2} + \frac{h}{2}\left(-\frac{g}{\ell}\sin(y_{n,1}) - \frac{g}{\ell}\sin(\overline{y}_{n+1,1})\right), \quad n = 0, 1, \ldots \end{aligned}$$

starting at $y_1 = \pi/4$ (initial angle), $y_2 = 0$ (initial velocity).

the modified Euler method in Julia

```
"""
    eulermodpend(p::Int,n::Int)

applies the modified Euler's method with n*p evaluations
on the interval [0,p*2*pi] on the pendulum problem.
"""
function eulermodpend(p::Int,n::Int)
    h = 2*p*pi/n
    (rt, ry1, ry2) = (zeros(n+1), zeros(n+1), zeros(n+1))
    (y1, y2) = (pi/4, 0)
    (rt[1], ry1[1], ry2[1]) = (0, y1, y2)
    for i=1:n
        t = i*h
        (y1a, y2a) = (y1 + h*y2, y2 + h*(-9.807*sin(y1)))
        (y1b, y2b) = (y1, y2)
        y1 = y1 + (h/2)*(y2a + y2b)
        y2 = y2 + (h/2)*(-9.807*sin(y1a)-9.807*sin(y1b))
        (rt[i+1], ry1[i+1], ry2[i+1]) = (t, y1, y2)
    end
    return (rt, ry1, ry2)
end
```

comparing the errors

```
$ julia eulerpend.jl
Running Euler's method ...
    1   0.00   7.853982e-01   0.000000e+00  0.00e+00
  101   6.28   5.533318e-01  -6.244064e+00  2.32e-01
  201  12.57  -3.787590e+01  -8.959735e+00  3.87e+01
  301  18.85  -9.276810e+01  -9.645784e+00  9.36e+01
Running the modified Euler method ...
    1   0.00   7.853982e-01   0.000000e+00  0.00e+00
  101   6.28   7.891436e-01  -3.622434e-01  3.75e-03
  201  12.57   7.800871e-01  -6.669493e-01  5.31e-03
  301  18.85   7.662499e-01  -9.095848e-01  1.91e-02
$
```

the pendulum with damping

The constant $d > 0$ takes friction into account:

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= -\frac{g}{\ell}\sin(y_1) - d\, y_2. \end{aligned}$$

Exercise 1:

1. Apply the modified Euler method for $\ell = 1$, $g = 9.807$, $d = 0.1$, for $y(0) = \pi/4$, $y'(0) = 0$, in the interval $[0, 6\pi]$ with $n = 500$.
2. Make a plot of the angles and the velocities.
3. Verify the correctness and run with higher values of n if needed.

the pendulum with damping and forcing

With damping, the pendulum comes to rest.

Adding $A\sin(t)$, for $A > 0$, keeps the pendulum moving:

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= -\frac{g}{\ell}\sin(y_1) - d\,y_2 + A\sin(t). \end{aligned}$$

Exercise 2:

1. Apply the modified Euler method for $\ell = 1$, $g = 9.807$, $d = 0.1$, for $y(0) = \pi/4$, $y'(0) = 0$, in the interval $[0, 6\pi]$ with $n = 500$. Use $A = 10$ for the amplitude of the forcing term.
2. Make a plot of the angles and the velocities.
3. Verify the correctness and run with higher values of n if needed.

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the cardioid

The cardioid is a curve in the plane, defined by

$$x(t) = (1 - \cos(t))\cos(t), \quad y(t) = (1 - \cos(t))\sin(t), \quad t \in [0, 2\pi].$$

It is the exact solution to the initial value problem

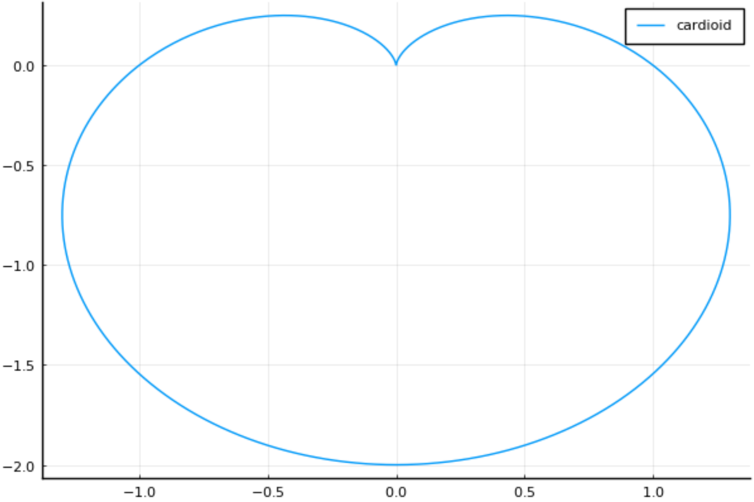
$$\left\{ \begin{array}{rcl} \dfrac{dx}{dt} & = & -y + \cos(t)\sin(t) \\ \dfrac{dy}{dt} & = & x + \sin^2(t) \end{array} \right. \qquad x(0) = 0, y(0) = 0.$$

The system written in matrix format:

$$\left[\begin{array}{c} x' \\ y' \end{array} \right] = \left[\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right] \left[\begin{array}{c} x \\ y \end{array} \right] + \left[\begin{array}{c} \cos(t)\sin(t) \\ \sin^2(t) \end{array} \right]$$

This is an example of a *linear system*.

the cardioid



apply the modified Euler method

Exercise 3:

Apply the modified Euler method to

$$\left\{ \begin{array}{rcl} \dfrac{dx}{dt} & = & -y + \cos(t)\sin(t) \\ \dfrac{dy}{dt} & = & x + \sin^2(t) \end{array} \right. \qquad x(0) = 0, y(0) = 0,$$

for $t \in [0, 2\pi]$.

Set h to be small enough so the plot of the computed points agrees with the plot of the exact solution.

another plane curve

Exercise 4: Consider the system

$$\left\{ \begin{array}{rcl} \dfrac{dx}{dt} & = & -y + 3\cos(3t)\cos(t) \\ \dfrac{dy}{dt} & = & x + 3\cos(3t)\sin(t) \end{array} \right. \qquad x(0) = 0, y(0) = 0.$$

1. Verify that $x(t) = \sin(3t)\cos(t)$, $y(t) = \sin(3t)\sin(t)$ is the exact solution.
2. Apply the modified Euler method to compute the solution curve, for $t \in [0, 2\pi]$.

   Set h to be small enough so the plot of the computed points agrees with the plot of the exact solution.

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linear systems

The *n*-th order differential equation

$$y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y^{(2)} + a_1 y^{(1)} + a_0 y^{(0)} = f$$

is equivalent to

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \\ \vdots \\ y_n' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ f \end{bmatrix}.$$

In short notation: $\mathbf{y}' = A\mathbf{y} + \mathbf{b}$ is a *linear system*.

solving a linear system

The linear system $\mathbf{y}' = A\mathbf{y} + \mathbf{b}$ is solved in two stages:

1. Find a solution $\mathbf{y}_h$ to the homogeneous system, with $\mathbf{b} = 0$.
2. Compute a particular solution $\mathbf{y}_p$, for a $\mathbf{b} \neq 0$.

The first-order equation

$$\frac{dy}{dx} = y, \quad y(0) = y_0$$

has the exact solution $y(x) = y_0 e^x$.

For the homogeneous linear system $\mathbf{y}' = A\mathbf{y}$, $\mathbf{y}(0) = \mathbf{y}_0$ is solved by

$$\mathbf{y}(x) = e^{Ax}\mathbf{y}(0), \quad e^{Ax} = I + Ax + \frac{(Ax)^2}{2!} + \frac{(Ax)^3}{3!} + \cdots$$

a spectral decomposition

A spectral decomposition of an n-by-n matrix A consists of

1. n eigenvalues λ_k (all *assumed* to be distinct), and
2. n corresponding eigenvectors $\mathbf{v}_k$, $k = 1, 2, \ldots, n$.

with $A\mathbf{v}_k = \lambda \mathbf{v}_k$, $k = 1, 2, \ldots, n$.

In matrix format $AV = V\Lambda$ or $A = V\Lambda V^{-1}$, with

$$\Lambda = \left[\begin{array}{cccc} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{array} \right] \quad \text{and} \quad V = \left[\begin{array}{cccc} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{array} \right].$$

We apply $A = V\Lambda V^{-1}$ to compute powers of A:

$$A^2 = (V\Lambda V^{-1})(V\Lambda V^{-1}) = V\Lambda^2 V^{-1}.$$

For any k, we have $A^k = V\Lambda^k V^{-1}$.

a general solution

For a homogeneous linear system $\mathbf{y}' = A\mathbf{y}$ with initial condition $\mathbf{y}(0) = \mathbf{y}_0$, has the general solution $\mathbf{y}(x) = e^{Ax}\mathbf{y}_0$.

We use $A = V\Lambda V^{-1}$ to rewrite $\mathbf{y}(x) = e^{Ax}\mathbf{y}_0$ as

$$\begin{array}{rcl} \mathbf{y}(x) & = & Ve^{\Lambda x}V^{-1}\mathbf{y}_0 \\ V^{-1}\mathbf{y}(x) & = & e^{\Lambda x}V^{-1}\mathbf{y}_0 \\ \mathbf{z}(x) & = & e^{\Lambda x}\mathbf{z}_0 \end{array}$$

A general solution is a linear combination of eigenvectors:

$$\begin{array}{rcl} \mathbf{y}(x) & = & V\mathbf{z}(x) \\ \mathbf{y}(x) & = & c_1 e^{\lambda_1 x}\mathbf{v}_1 + c_2 e^{\lambda_2 x}\mathbf{v}_2 + \cdots + c_n e^{\lambda_n x}\mathbf{v}_n \end{array}$$

The *complex* eigenvalues determine whether the solution is periodic, diverging, or decaying.

the linear system of the cardioid

Exercise 5:

Consider the linear system for the cardioid:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \cos(t)\sin(t) \\ \sin^2(t) \end{bmatrix}$$

1. Use the eigenvalues and eigenvectors to write the general solution for the homogeneous part of this problem.
2. Does the general form show that the solution is periodic? Justify your answer.

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rabbits and foxes

The growth of the rabbit population $r(t)$ is modeled via

$$\frac{dr(t)}{dt} = a\,r(t), \quad a > 0, \quad a \text{ is the growth rate.}$$

The decay of the foxes population $f(t)$ is modeled via

$$\frac{df(t)}{dt} = c\,f(t), \quad c < 0, \quad c \text{ is the decay rate.}$$

About the growth and decay rates:

- without predators, the rabbit population grows exponentially;
- without prey, the fox population decays exponentially.

two nonlinear ordinary differential equations

A model for the evolution of the two population sizes over time t is a system of two nonlinear ordinary differential equations:

$$\left\{ \begin{array}{rcll} \dfrac{dr(t)}{dt} & = & a\,r(t) + b\,r(t)f(t), & a > 0, b < 0, \\ \dfrac{df(t)}{dt} & = & c\,f(t) + d\,r(t)f(t), & c < 0, d > 0. \end{array} \right.$$

The product term $r(t)f(t)$ corrects the exponential growth and decay:

- $b < 0$ as a rabbit-fox encounter is negative for the rabbit,
- $d > 0$ as a rabbit-fox encounter is positive for the fox.

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application of the modified Euler method

To apply the modified Euler method, we use a standard notation. Let $y_1(t) = r(t)$ and $y_2(t) = f(t)$. Then the system is

$$\begin{array}{rcl} y_1' & = & F_1(t, y_1, y_2), \quad F_1 = ay_1 + by_1 y_2, \\ y_2' & = & F_2(t, y_1, y_2), \quad F_2 = cy_2 + dy_1 y_2. \end{array}$$

For $t_n = nh$, $h > 0$, we compute $y_{n,1} \approx y_1(t_n)$, $y_{n,2} \approx y_2(t_n)$.

1. Predict with Euler's method:

$$\begin{array}{rcl} \overline{y}_{n+1,1} & = & y_{n,1} + hF_1(t_n, y_{n,1}, y_{n,2}), \\ \overline{y}_{n+1,2} & = & y_{n,2} + hF_2(t_n, y_{n,1}, y_{n,2}). \end{array}$$

2. Correct the predicted values:

$$\begin{array}{rcl} y_{n+1,1} & = & y_{n,1} + (h/2)(F_1(t_n, y_{n,1}, y_{n,2}) + F_1(t_n, \overline{y}_{n+1,1}, \overline{y}_{n+1,2})), \\ y_{n+1,2} & = & y_{n,2} + (h/2)(F_2(t_n, y_{n,1}, y_{n,2}) + F_2(t_n, \overline{y}_{n+1,1}, \overline{y}_{n+1,2})). \end{array}$$

running the simulation

We need to specify numerical values for the simulation.

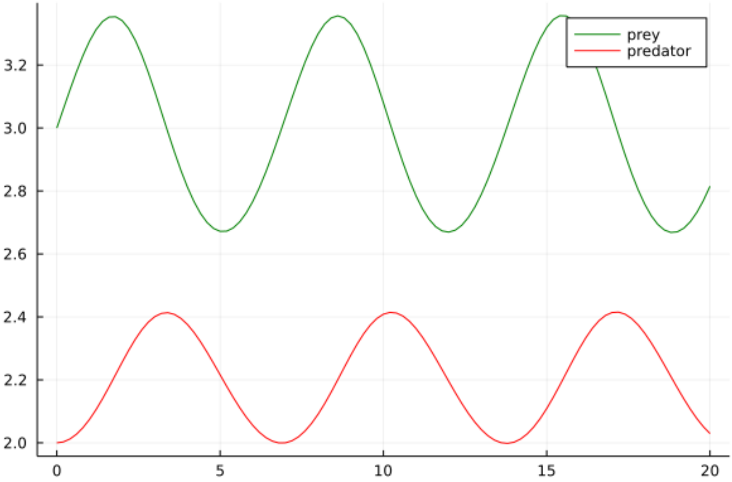
1. Values of the four parameters in the system:

$$\begin{array}{rcll} a & = & 1.1, & \text{growth rate of the prey} \\ b & = & -0.5, & \text{correction to prey from encounter with predator} \\ c & = & -0.75, & \text{decay rate of the predator} \\ d & = & 0.25, & \text{correction to predator from encounter with prey} \end{array}$$

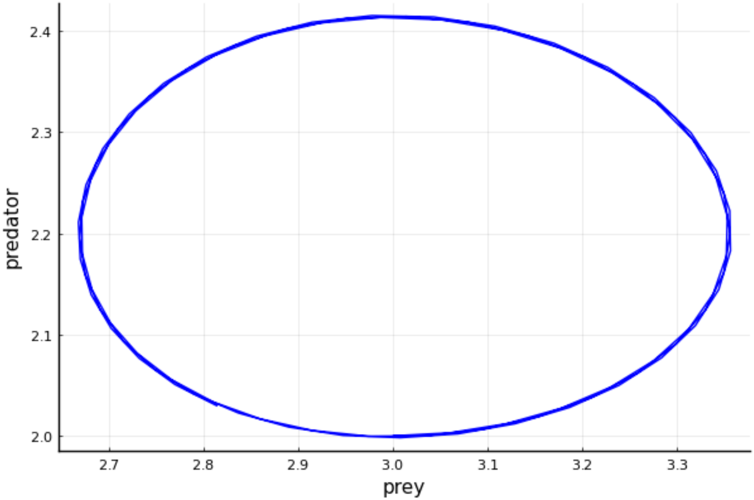
2. The initial values for the populations: $y_1(0) = 3$, $y_2(0) = 2$.
3. The time interval and the number of steps: $T = 100$, $n = 20$.

See the posted Julia program and Jupyter notebook.

plots of the population sizes



the phase portrait



two questions on the simulation

Exercise 6:

For the predator-prey simulation use the posted Jupyter notebook to answer the following two questions:

1. Determine experimentally the period of the simulation?

   The period is when the phase portrait completes one cycle.

2. What is the critical value for the step size h?

   The critical value is the optimal value:

   - we do the smallest number of steps,
   - and still obtain correct results.