Review for the First Midterm Exam

1. The First Midterm Exam
   - on Friday 14 October at 10am
   - skipping policy

2. Some Questions
   - floating-point numbers and arithmetic
   - root finding
   - solving linear systems
   - interpolation and data fitting

MCS 471 Lecture 22
Numerical Analysis
Jan Verschelde, 12 October 2022

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The First Midterm Exam

- Friday 14 October, at 10am online.
- You must decide if you take the exam by noon Thursday (13 Oct), send an email to `janv@uic.edu` if you will do the exam.
- The exam must be solved individually.
  Submitting materials retrieved from the internet is plagiarism.
- Solutions must be in a Jupyter notebook, with a Julia kernel.
- Answers must be submitted before on on 10:50pm.
- Submit to gradescope.
- Because of the skipping policy, there is no makeup exam.

Topics on the First Midterm Exam

- The midterm convers lectures 1 to 21.
  1. Floating-point numbers are arithmetic.
  2. Root finding methods, the golden section search.
  3. Solving linear systems.
  4. Interpolation and data fitting.
- Focus on numerical analysis concepts, not on Julia programming.
- Questions on this review are representative,
  but the list is by no means exhaustive.
- Review the projects, quizzes, and homework problems.

overview of algorithms

1. the bisection method
2. secant, regula falsi, and Newton's method
3. the golden section search method
4. LU factorization with row pivoting
5. the methods of Jacobi, Gauss-Seidel, successive over-relaxation
6. Cholesky factorization; the conjugate gradient method
7. Lagrange, Neville, and Newton interpolation
8. Chebyshev interpolation
9. rational approximations
10. cubic splines; Bézier curves
11. QR factorization via Gram-Schmidt and Householder
12. the Generalized Minimum Residual Method
13. the power method; Gauss-Newton, Levenberg-Marquardt

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policy on skipping a midterm exam

Please note the policy on skipping the exam:

If an exam is missed, then greater weight will be placed on the final exam, especially on the material covered on the missing exam.

- What this means is that if you decide not to take one midterm exam, your final exam will count for one hundred points more.
- What it does NOT mean is that you can drop the score of a midterm exam.

If you take the midterm, then your score counts.

Please be prepared when you show up for the exam.

Showing up for the exam means that you email `janv@uic.edu` before noon on Thursday 13 October that you will take the exam.

Without sending an email you will not receive the questions.

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1. floating-point numbers and arithmetic

Consider a floating-point number system with base 10,
3 digits in the fraction and exponents ranging between -11 and $+11$.

1. What is the machine precision of this number system?
2. Calculate how many positive numbers we can represent in this number system.
3. Represent the numbers 12345 and 218.37 in this number system. Show how to calculate the sum.

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2. root finding

Consider the equation $f(x) = e^x - 4x^3$.
Find an approximation for the root of f,

1. using four steps of the bisection method on the interval $[0, 1]$;
2. using four steps of the secant method starting at $x_0 = 0$ and $x_1 = 1$;
3. using four steps of Newton's method starting at $x = 1$.

3. Newton's method

Use Newton's method to find the cube root of 27.13, with four decimal places of accuracy.

4. multiple roots

The following is the output of a Newton's method running on $f(x) = (x-1)^m$:

x	|dx|	|f(x)|
9.25000000000528e-01	2.50e-02	3.16e-05
9.43749999999878e-01	1.87e-02	1.00e-05
9.57812499999771e-01	1.41e-02	3.17e-06
9.68359375005512e-01	1.05e-02	1.00e-06
9.76269531245741 5e-01	7.91e-03	3.17e-07
9.82202148439873 0e-01	5.93e-03	1.00e-07
9.86651611338519 6e-01	4.45e-03	3.17e-08
9.89988708494472 2e-01	3.34e-03	1.00e-08
9.92491531357865 1e-01	2.50e-03	3.18e-09
9.94368648520889 1e-01	1.88e-03	1.01e-09
9.95776486750578 5e-01	1.41e-03	3.18e-10
9.96832366697024 5e-01	1.06e-03	1.01e-10
9.97624273890275 7e-01	7.92e-04	3.19e-11
9.98218200170579 4e-01	5.94e-04	1.01e-11
9.98663638501326 7e-01	4.45e-04	3.19e-12
9.98997685929897 7e-01	3.34e-04	1.01e-12
9.99248378594142 4e-01	2.51e-04	3.19e-13
9.99436240861424 2e-01	1.88e-04	1.01e-13
9.99577515699539 1e-01	1.41e-04	3.21e-14
9.99683885052209 8e-01	1.06e-04	9.77e-15

What is m?

5. fixed point iterations

The following fixed-point iterations all have the same fixed point, i.e.: 5,

1. $x_{k+1} = \frac{1}{5}x_k^2$, $k = 0, 1, \ldots$
2. $x_{k+1} = \sqrt{5x_k}$, $k = 0, 1, \ldots$
3. $x_{k+1} = \frac{x_k^2}{2x_k - 5}$, $k = 0, 1, \ldots$

For each of the three fixed-point iterations, make a cobweb picture illustrating the convergence (or divergence), starting at $x_0 = 4$. Compute the convergence (or divergence) rate for each iteration. Which iteration is best?

6. the golden section search

Suppose we have an interval enclosing
the minimum of a unimodal function of length equal to one.

How many function evaluations does the golden section search need to approximate the minimum with two decimal places?

Justify your answer!

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7. row reduction

Consider the linear system

$$\left\{ \begin{array}{l} 2.000x_1 + 5.000x_2 + 3.000x_3 = 10.00 \\ 2.000x_1 + 8.000x_2 + 1.000x_3 = 11.00 \\ 3.000x_1 + 4.000x_2 + 3.000x_3 = 10.00 \end{array} \right.$$

1. Use Gaussian elimination to compute an LU decomposition without and without row pivoting.
2. Solve the system two times, using the two LU decompositions obtained from above.
3. Compute the determinant using the two LU decompositions obtained from above.
4. Use the LU decomposition to compute A^{-1} and $\mathrm{cond}(A)$ with $||.||_1$.

8. factoring a tridiagonal matrix

A tridiagonal matrix A has a special LU decomposition

$$\begin{bmatrix} a_1 & b_1 & & \\ c_2 & a_2 & b_2 & \\ & c_3 & a_3 & b_3 \\ & & c_4 & a_4 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ \ell_1 & 1 & & \\ & \ell_2 & 1 & \\ & & \ell_3 & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & & \\ & u_{22} & u_{32} & \\ & & u_{33} & u_{34} \\ & & & u_{44} \end{bmatrix}$$

assuming none of the diagonal elements in the reduction turns zero.

Derive the formulas to compute this decomposition.
Write the algorithm for general tridiagonal matrices like A.

9. the numerical condition of a linear system

Let $\overline{\mathbf{x}}$ be a numerically computed solution of the linear system $A\mathbf{x} = \mathbf{b}$, computed with a precision equal to 32 decimal places.

1. How to compute the backward error for this problem?
2. Suppose the condition number of the matrix A is 10^{12}.
   How many decimal places in $\overline{\mathbf{x}}$ are correct? Justify your answer.
3. Describe a computation to estimate the forward error.

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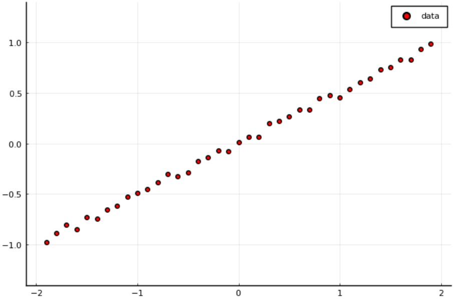
10. polynomial interpolation

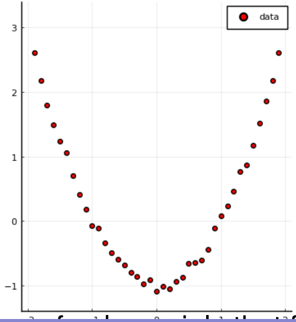
Consider the polynomial $p(x) = x^2 - 5x + 1$.

1. Construct the Newton form of $p(x)$ by divided differences, using the points $(x_i, p(x_i))$, with $x_i = i$, for $i = 0, 1, 2, 3$.
2. Explain why the last element f_{0123} in the table of divided differences you constructed above is (or should have been) zero.
3. Apply Neville's algorithm to evaluate the interpolating polynomial at 0.5.
4. Approximate $p(x)$ with the linear function that minimizes the squares of the errors, using the points $(x_i, p(x_i))$, with $x_i = i$, for $i = 0, 1, 2, 3$.

11. fitting data

Consider the data





12. the power method

Can the convergence of the power method be quadratic?
Justify your answer.

13. there is always more ...