Shooting Methods

1. Boundary Value Problems
   - Dirichlet and Neumann conditions
   - an application of the shooting method
   - linear problems

2. Nonlinear Problems
   - an example with Dirichlet conditions
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Dirichlet and Neumann conditions

Consider the boundary value problem

\[ \frac{d^2 u}{dx^2} = f(x, u), \quad x \in [a, b]. \]

We distinguish between two types of boundary conditions:

1. **Dirichlet** imposes conditions on values at end points.
   
   \[ u(a) = A \quad \text{and} \quad u(b) = B. \]

2. **Neumann** imposes conditions on derivatives at the ends.
   
   \[ u'(a) = A \quad \text{and} \quad u'(b) = B. \]
Consider
\[ \frac{d^2 u}{dx^2} = u(x), \]
with conditions on the boundaries
\[ u'(1) = 1.17520 \quad \text{and} \quad u'(3) = 10.0179. \]

To reduce this to an initial value problem, we need to know \( u(1) \).
\[ p(u(1)) = u'(3) = 10.0179 \]

We predict the value for \( u(1) \) to reach \( u'(3) \).
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application of shooting

We solve two initial value problems:

- $u(1) = 1$ leads to $u'(3) = 8.04819$
- $u(1) = 2$ leads to $u'(3) = 11.6751$

Since $u'(3) = 10.0179$, with $u(1) = 1$ we undershot the solution, and with $u(1) = 2$ we overshot the solution.

Let us predict the next value for $u(1)$ by linear interpolation:

$$p(z) = \left(\frac{z - 1}{2 - 1}\right) 11.6751 + \left(\frac{z - 2}{1 - 2}\right) 8.04819$$

$$= 3.62691z + 4.42128.$$
interpolation continued

\[ p(z) = \left( \frac{z - 1}{2 - 1} \right) 11.6751 + \left( \frac{z - 2}{1 - 2} \right) 8.04819 \]
\[ = 3.62691z + 4.42128. \]

We have to find \( z \) such that \( p(z) = u'(3) = 10.0179: \)

\[ z = \frac{10.0179 - 4.42128}{3.62619} = 1.54308. \]

The next IVP will be solved with \( u'(1) = 1.54308. \)
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Linear Ordinary Differential Equations

Consider the ordinary differential equation

\[ u'' + Fu' + Gu = H, \]

with constant coefficients \( F, \) \( G, \) and \( H. \)

If \( u_1 \) and \( u_2 \) are solutions, then also \( \frac{c_1 u_1 + c_2 u_2}{c_1 + c_2} \) is a solution for any two constants \( c_1 \) and \( c_2. \)

Let us verify:

\[
\frac{c_1 u_1'' + c_2 u_2''}{c_1 + c_2} + F \frac{c_1 u_1' + c_2 u_2'}{c_1 + c_2} + G \frac{c_1 u_1 + c_2 u_2}{c_1 + c_2} \\
= \frac{c_1 (u_1'' + Fu_1' + Gu_1) + c_2 (u_2'' + Fu_2' + Gu_2)}{c_1 + c_2} \\
= \frac{c_1 H + c_2 H}{c_1 + c_2} = H.
\]
the example revisited

Given is $u'(1) = 1.17520$ and $u'(3) = 10.0179$.

- $u(1) = 1$ leads to $u'(3) = 8.04819$
- $u(1) = 2$ leads to $u'(3) = 11.6751$

The equations $u'' = u$ is linear, which means that for any two solutions $u_1$, $u_2$, also $\frac{c_1 u_1 + c_2 u_2}{c_1 + c_2}$ is a solution.

$$u(x) = \frac{c_1 u_1 + c_2 u_2}{c_1 + c_2} \Rightarrow u'(x) = \frac{c_1}{c_1 + c_2} u'_1 + \frac{c_2}{c_1 + c_2} u'_2.$$ 

Let $\gamma_1 = c_1/(c_1 + c_2)$ and $\gamma_2 = c_2/(c_1 + c_2)$, then:

$$u'(1) = 1.17520 = \gamma_1 u'_1(1) + \gamma_2 u'_2(1)$$
$$u'(3) = 10.0179 = \gamma_1 u'_1(3) + \gamma_2 u'_2(3)$$

We have computed two solutions: $u'_1(3) = 8.04819$ and $u'_2(3) = 11.6751$. Moreover: $u'_1(1) = 1.17520 = u'_2(1)$. 
a linear system

\[ \begin{align*}
  u'(1) &= 1.17520 &= \gamma_1 u'_1(1) + \gamma_2 u'_2(1) \\
  u'(3) &= 10.0179 &= \gamma_1 u'_1(3) + \gamma_2 u'_2(3)
\end{align*} \]

We have computed two solutions: \( u'_1(3) = 8.04819 \) and \( u'_2(3) = 11.6751 \). Moreover: \( u'_1(1) = 1.17520 = u'_2(1) \).

Thus we can solve the system for \( \gamma_1 \) and \( \gamma_2 \):

\[
\begin{bmatrix}
  1.17520 & 1.17520 \\
  8.04819 & 11.6751
\end{bmatrix}
\begin{bmatrix}
  \gamma_1 \\
  \gamma_2
\end{bmatrix} =
\begin{bmatrix}
  1.17502 \\
  10.0179
\end{bmatrix}
\]

This implies that for linear problems, solving two initial value problems suffices to find the solution to the boundary value problem.
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a nonlinear boundary value problem

Consider

$$\frac{d^2 u}{dx^2} - \left(1 - \frac{x}{5}\right) u(x) \frac{du}{dx} = x, \quad x \in [1, 3],$$

with conditions on the boundaries

$$u(1) = 2 \quad \text{and} \quad u(3) = -1.$$ 

Observe that the coefficients depend on $x$.

This is nonlinear boundary value problem.
solving a nonlinear boundary value problem

Consider
\[
\frac{d^2 u}{dx^2} - \left(1 - \frac{x}{5}\right) u(x) \frac{du}{dx} = x, \quad x \in [1, 3],
\]
with conditions on the boundaries
\[
u(1) = 2 \quad \text{and} \quad u(3) = -1.
\]

We solve this problem by shooting, with a reduction to an initial value problem.
We know \(u(1) = 2\), but what is \(u'(1)\)?

\[
p(u'(1)) = u(3) = -1
\]

We predict the value for \(u'(1)\) to reach \(-1\).