Shooting Methods

1. Boundary Value Problems
   - a falling object
   - shooting
   - interpolation

2. Linear Problems
   - equations with constant coefficients
   - Dirichlet and Neumann conditions

3. Nonlinear Problems
   - an example with Dirichlet conditions
   - the pendulum as a nonlinear BVP

MCS 471 Lecture 33
Numerical Analysis
Jan Verschelde, 7 November 2022
Shooting Methods

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a falling object

An object falls to the ground from a height of 30 meters. Its height as a function in time $t$ is represented by $y(t)$. At time $t = 0$, we then write $y(0) = 30$.

Assuming the mass of the object equals one, by Newton’s law, its acceleration is determined by

$$\frac{d^2y}{dt^2} = -g, \quad g = 9.81.$$ 

Not knowing the initial velocity, we impose that at time $t = 4$, the object hits the ground.

So we have an additional boundary condition: $y(4) = 0$. 

The falling object problem is now a boundary value problem:

\[ y''(t) = \frac{d^2y}{dt^2} = -g, \quad g = 9.81, \quad y(0) = 30, \quad y(4) = 0. \]

Its solution is obtained by integrating twice:

\[ y'(t) = \int (-g)dt = -gt + v_0, \quad y(t) = \int (-gt + v_0)dt = -\frac{gt^2}{2} + v_0 t + y_0. \]

To compute the two unknown constants \( v_0 \) and \( y_0 \) we use the boundary conditions: \( y(0) = 30 = y_0 \) and

\[ y(4) = 0 : \quad -g\frac{16}{2} + v_0 4 + 30 = 0 \Rightarrow v_0 = \left( +g\frac{16}{2} - 30 \right) / 4 = 12.12. \]
A boundary value problem of order two has the form

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right), \quad x \in [a, b],$$

with conditions on the boundaries

$$y(a) = y_a \quad \text{and} \quad y(b) = y_b.$$  

The corresponding initial value problem has conditions

$$y(a) = y_a \quad \text{and} \quad y'(a) = ?.$$  

For this to be an initial value problem, we must know $y'(a)$.

*Can we reduce this to an initial value problem?*
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reduction to an initial value problem

Given at the left are the boundary values $y(a)$ and $y(b)$. We want to compute the trajectory from $(a, y(a))$ to $(b, y(b))$.

If we knew $y'(a)$, then we would have an initial value problem.
the shooting method

We take $\bar{y}'(a)$ as a guess for $y'(a)$ and solve the initial value problem with $y(a)$ and $y'(a)$.

We end up with $\bar{y}(b) < y(b)$, we undershot.
the shooting method, try again

We take $\bar{y}'(a)$ as a guess for $y'(a)$ and solve the initial value problem with $y(a)$ and $y'(a)$.

We end up with $\bar{y}(b) > y(b)$, we overshot.
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interpolation

- We used two different values for $\bar{y}'(a)$, let us call them $Y_1$ and $Y_2$.
- We obtained two different values for $y(b)$: say $F_1$ and $F_2$.

View $F_1$ as a function of $Y_1$ and $F_2$ as a function of $Y_2$:

$$F(Y_1) = F_1 \quad \text{and} \quad F(Y_2) = F_2.$$ 

We interpolate (remember Lagrange):

$$p(Y) = \left( \frac{Y_2 - Y}{Y_2 - Y_1} \right) F_1 + \left( \frac{Y_1 - Y}{Y_1 - Y_2} \right) F_2.$$ 

Observe: $p(Y_1) = F_1$ and $p(Y_2) = F_2$. 
get ready for the next shot

To solve
\[ \frac{d^2y}{dx^2} = f \left( x, y, \frac{dy}{dx} \right), \quad x \in [a, b], \quad y(a) = y_a, \quad y(b) = y_b, \]
as an IVP, we tried \( Y_1 \) and \( Y_2 \) as values for \( y'(a) \)
and ended up with \( F_1 \) and \( F_2 \) for \( y_b \), as written by the polynomial
\[ p(Y) = \left( \frac{Y_2 - Y}{Y_2 - Y_1} \right) F_1 + \left( \frac{Y_1 - Y}{Y_1 - Y_2} \right) F_2. \]

For the next shot for \( y'(a) \), we compute \( Y_3 \) via
\[ p(Y_3) = u(b) \quad \text{and solve} \quad \left( \frac{Y_2 - Y_3}{Y_2 - Y_1} \right) F_1 + \left( \frac{Y_1 - Y_3}{Y_1 - Y_2} \right) F_2 = u(b). \]
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linear ordinary differential equations

Consider the ordinary differential equation

\[ y'' + Fy' + Gy = H, \]

with constant coefficients \( F, \) \( G, \) and \( H. \)

If \( y_1 \) and \( y_2 \) are solutions, then also \( \frac{c_1 y_1 + c_2 y_2}{c_1 + c_2} \) is a solution for any two constants \( c_1 \) and \( c_2. \)

Let us verify:

\[
\frac{c_1 y_1'' + c_2 y_2''}{c_1 + c_2} + F \frac{c_1 y_1' + c_2 y_2'}{c_1 + c_2} + G \frac{c_1 y_1 + c_2 y_2}{c_1 + c_2} \\
= \frac{c_1 (y_1'' + Fy_1' + Gy_1) + c_2 (y_2'' + Fy_2' + Gy_2)}{c_1 + c_2} = \frac{c_1 H + c_2 H}{c_1 + c_2} = H.
\]
the falling object problem again

The falling object problem

\[ y''(t) = \frac{d^2 y}{d t^2} = -g, \quad g = 9.81, \quad y(0) = 30, \quad y(4) = 0 \]

is a linear problem, with exact solution \(-g \frac{t^2}{2} + 12.12t + 30\).

Instead of a numerical solver, we can use the exact solution for when we have to compute \(y(4)\) given a value for \(y'(0)\).

**Exercise 1:**
Apply the shooting method to the falling object problem above, use \(Y_1 = 10\) and \(Y_2 = 14\) for the values for \(y'(0)\).
You may use the exact solution instead of a numerical solver.
Exercise 2:
Consider the boundary value problem:

\[ y''(t) = \frac{d^2y}{dt^2} = -g, \quad g = 9.81, \quad y(0) = 30, \quad y(4) = 0. \]

Instead of using the exact solution of Exercise 1, apply \texttt{RK45} of the \texttt{integrate} module of \texttt{SciPy} in the shooting method, using \( Y_1 = 10 \) and \( Y_2 = 14 \) for the values of \( y'(0) \).
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Dirichlet and Neumann conditions

Consider the boundary value problem

\[ \frac{d^2 y}{dx^2} = f \left( x, y, \frac{dy}{dx} \right), \quad x \in [a, b]. \]

We distinguish between two "pure" types of boundary conditions:

1. **Dirichlet** imposes conditions on values at end points.
   \[ y(a) = A \quad \text{and} \quad y(b) = B. \]

2. **Neumann** imposes conditions on derivatives at the ends.
   \[ y'(a) = A \quad \text{and} \quad y'(b) = B. \]

Conditions of "mixed" type combine values and derivatives.
the shooting method for Neumann conditions

We take $\bar{y}(a)$ as a guess for $y(a)$ and solve the initial value problem with $\bar{y}(a)$ and $y'(a)$.

1. Use two different values $Y_1$ and $Y_2$ for $\bar{y}(a)$ and label the corresponding values for $y'(b)$ as $F_1$ and $F_2$.

2. Let $p(Y)$ be the interpolating polynomial:

$$p(Y_1) = F_1 \quad \text{and} \quad p(Y_2) = F_2.$$ 

3. Determine $Y_3$ as the solution of $p(Y_3) = y'(b)$. 

Let $y(1) = 1$ and $y'(3) = 10.0179$ be two boundary conditions.

Suppose we solved two initial value problems:

- Taking $Y_1 = 1$ for $y(1)$ leads to $F_1 = 8.04819$ for $y'(3)$.
- Taking $Y_2 = 2$ for $y(1)$ leads to $F_3 = 11.6751$ for $y'(3)$.

Since $y'(3) = 10.0179$, with $Y_1 = 1$ we undershot the solution, and with $Y_2 = 2$ we overshot the solution.

Let us predict the next value for $y(1)$ by linear interpolation:

$$p(Y) = \left(\frac{Y - 1}{2 - 1}\right)11.6751 + \left(\frac{Y - 2}{1 - 2}\right)8.04819$$

$$= 3.62691Y + 4.42128.$$
interpolation continued

\[ p(Y) = \left( \frac{Y - 1}{2 - 1} \right) 11.6751 + \left( \frac{Y - 2}{1 - 2} \right) 8.04819 \]
\[ = 3.62691 Y + 4.42128. \]

We have to find \( Y_3 \) such that \( p(Y_3) = y'(3) = 10.0179: \)

\[ Y_3 = \frac{10.0179 - 4.42128}{3.62619} = 1.54308. \]

The next IVP will be solved with 1.54308 as the value for \( y(1). \)
Consider $y'' = y$, with $y'(1) = 1.17520$ and $y'(3) = 10.0179$.

- $y(1) = 1$ leads to $y'(3) = 8.04819$
- $y(1) = 2$ leads to $y'(3) = 11.6751$

The equation $y'' = y$ is linear, which means that for any two solutions $y_1$, $y_2$, also $\frac{c_1 y_1 + c_2 y_2}{c_1 + c_2}$ is a solution.

$$y(x) = \frac{c_1 y_1 + c_2 y_2}{c_1 + c_2} \quad \Rightarrow \quad y'(x) = \frac{c_1}{c_1 + c_2} y_1' + \frac{c_2}{c_1 + c_2} y_2'.$$

Let $\gamma_1 = \frac{c_1}{c_1 + c_2}$ and $\gamma_2 = \frac{c_2}{c_1 + c_2}$, then:

$$y'(1) = 1.17520 = \gamma_1 y_1'(1) + \gamma_2 y_2'(1)$$
$$y'(3) = 10.0179 = \gamma_1 y_1'(3) + \gamma_2 y_2'(3)$$
a linear system

\[
\begin{align*}
y'(1) &= 1.17520 = \gamma_1 y_1'(1) + \gamma_2 y_2'(1) \\
y'(3) &= 10.0179 = \gamma_1 y_1'(3) + \gamma_2 y_2'(3)
\end{align*}
\]

We have two computed solutions: \( y_1'(3) = 8.04819 \) and \( y_2'(3) = 11.6751 \). Moreover: \( y_1'(1) = 1.17520 = y_2'(1) \).

Thus we can solve the system for \( \gamma_1 \) and \( \gamma_2 \):

\[
\begin{bmatrix} 1.17520 & 1.17520 \\ 8.04819 & 11.6751 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} = \begin{bmatrix} 1.17502 \\ 10.0179 \end{bmatrix}
\]

This example shows that for linear problems, solving two initial value problems suffices to find the solution to the boundary value problem.
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Consider
\[
\frac{d^2 y}{dx^2} - \left(1 - \frac{x}{5}\right) y(x) \frac{dy}{dx} = x, \quad x \in [1, 3],
\]
with conditions on the boundaries
\[
y(1) = 2 \quad \text{and} \quad y(3) = -1.
\]
Observe that the coefficients depend on $x$.

This is a \textit{nonlinear} boundary value problem.

Linear problems can be solved with linear interpolation. For nonlinear problems, higher degree interpolation is needed.
solving a nonlinear boundary value problem

Consider

\[ \frac{d^2 y}{dx^2} - \left( 1 - \frac{x}{5} \right) y(x) \frac{dy}{dx} = x, \quad x \in [1, 3], \]

with conditions on the boundaries

\[ y(1) = 2 \quad \text{and} \quad y(3) = -1. \]

We solve this problem by shooting, with a reduction to an initial value problem. We know \( y(1) = 2 \), but what is \( y'(1) \)?

\[ p(y'(1)) = y(3) = -1 \]

We predict the value for \( y'(1) \) to reach \(-1\).
Exercise 3:
Consider the boundary value problem

\[ y''(t) + t y'(t) - y(t) = t^2, \quad y(0) = 0, \quad y(5) = 20. \]

Using 1 and 2 as guesses for \( y'(0) \) gave respectively 19.47 and 24.47 for \( y(5) \).

1. What is your next guess \( Y_3 \) in the shooting method?
2. Use \texttt{SciPy.integrate.RK45} to compute the value for \( y(5) \) when solving the initial value problem with \( y(0) = Y_3 \).
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the pendulum as a nonlinear BVP

A model of a pendulum, without damping, without forcing:

\[
\frac{d^2y}{dt^2} = -g \sin(y),
\]

where \( g \) is the constant approximately equal to 9.807. Formulated as a Boundary Value Problem, we add two conditions:

\[
y(0) = \pi/6, \quad y(2\pi) = \pi/6.
\]
with initial velocity $y'(0) = 0$
the third trajectory
shooting with linear interpolation

After $y'(0) = 0$, the second initial velocity is $y'(0) = 1$.

The end angles are the values of the interpolating polynomial $p$:

\[
p(0) = 0.463 \\
p(1) = 0.605
\]

The interpolating polynomial is

\[
p(z) = \left( \frac{z - 1}{0 - 1} \right) 0.463 + \left( \frac{z - 0}{1 - 0} \right) 0.605.
\]

Solving $p(z) - \pi/6 = 0$ gives $z = 0.428$. 
shooting with quadratic interpolation

After $y'(0) = 0$, the second initial velocity is $y'(0) = 1$.

The end angles are the values of the interpolating polynomial $q$:

$$q(0) = 0.463, \quad q(1) = 0.605, \quad q(0.428) = 0.531.$$  

The interpolating polynomial is

$$q(z) = \left( \frac{z - 1}{0 - 1} \right) \left( \frac{z - 0.428}{0 - 0.428} \right) 0.463$$
$$+ \left( \frac{z - 0}{1 - 0} \right) \left( \frac{z - 0.428}{1 - 0.428} \right) 0.605$$
$$+ \left( \frac{z - 0}{0.428 - 0} \right) \left( \frac{z - 1}{0.428 - 1} \right) 0.531.$$  

Solving $q(z) - \pi/6 = 0$ gives $z = 0.379$ and $z = 5.37$.  
Take the closest solution to the previous initial velocity: $z = 0.379$.  

Numerical Analysis (MCS 471)
the pendulum with damping and forcing

The extended model of the pendulum adds damping and forcing:

\[
\frac{d^2 y}{dt^2} + A \frac{dy}{dt} + g \sin(y) = B \sin(t), \quad A = \frac{1}{10}, \quad B = 10,
\]

with boundary values \( y(0) = \pi/6, \quad y(2\pi) = \pi/6. \)

Exercise 4: Apply the shooting method to this problem in two ways:

1. With linear extrapolation, using the last two guesses for \( y'(0) \).
2. With quadratic extrapolation, using the last three guesses.

Organize your computations in a systematic manner.

Do you observe a difference between linear and quadratic extrapolation in the number of shooting steps needed to obtain correct values for \( y'(0) \)?