Shooting Methods

1. Boundary Value Problems
   - a falling object
   - shooting
   - interpolation
2. Linear Problems
   - equations with constant coefficients
   - Dirichlet and Neumann conditions
3. Nonlinear Problems
   - an example with Dirichlet conditions
   - the pendulum as a nonlinear BVP

MCS 471 Lecture 33
Numerical Analysis
Jan Verschelde, 7 November 2022

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a falling object

An object falls to the ground from a height of 30 meters.

Its height as a function in time t is represented by $y(t)$.

At time $t = 0$, we then write $y(0) = 30$.

Assuming the mass of the object equals one,
by Newton's law, its acceleration is determined by

$$\frac{d^2 y}{d t^2} = -g, \quad g = 9.81.$$

Not knowing the initial velocity, we impose that at time $t = 4$,
the object hits the ground.

So we have an additional boundary condition: $y(4) = 0$.

a boundary value problem

The falling object problem is now a boundary value problem:

$$y''(t) = \frac{d^2 y}{d\,t^2} = -g, \quad g = 9.81, \quad y(0) = 30, \quad y(4) = 0.$$

Its solution is obtained by integrating twice:

$$y'(t) = \int (-g) dt = -g\,t + v_0, \quad y(t) = \int (-g\,t + v_0) dt = -g\frac{t^2}{2} + v_0 t + y_0.$$

To compute the two unknown constants v_0 and y_0
we use the boundary conditions: $y(0) = 30 = y_0$ and

$$y(4) = 0: \ -g\frac{16}{2} + v_0 4 + 30 = 0 \Rightarrow v_0 = \left(+g\frac{16}{2} - 30 \right) /4 = 12.12.$$

a general boundary value problem of order two

A boundary value problem of order two has the form

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right), \quad x \in [a, b],$$

with conditions on the boundaries

$$y(a) = y_a \quad \text{and} \quad y(b) = y_b.$$

The corresponding initial value problem has conditions

$$y(a) = y_a \quad \text{and} \quad y'(a) = ?.$$

For this to be an initial value problem, we must know $y'(a)$.

Can we reduce this to an initial value problem?

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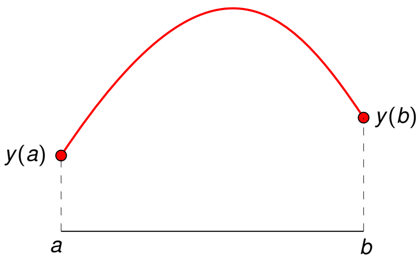
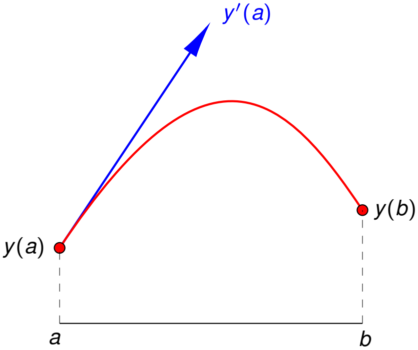
reduction to an initial value problem

Given at the left are the boundary values $y(a)$ and $y(b)$.
We want to compute the trajectory from $(a, y(a))$ to $(b, y(b))$.

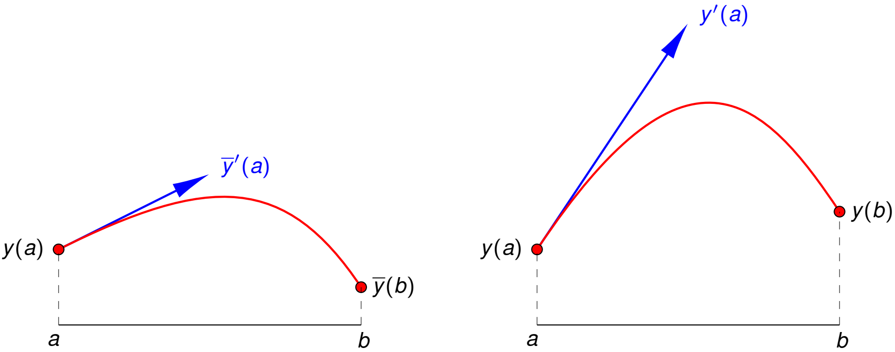




If we knew $y'(a)$, then we would have an initial value problem.

the shooting method

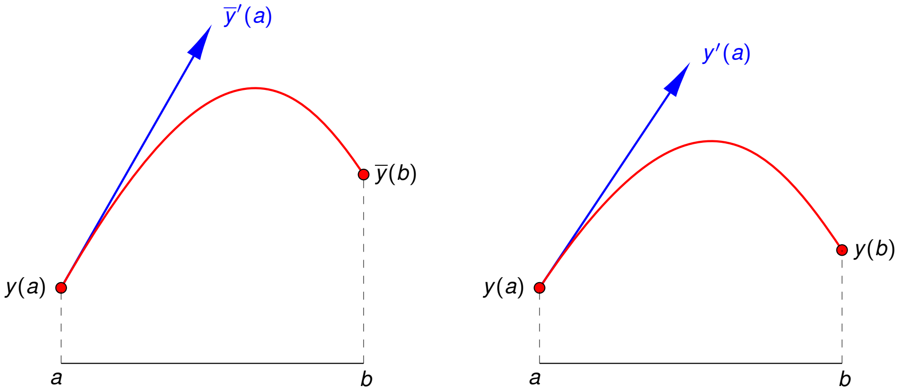
We take $\overline{y}'(a)$ as a guess for $y'(a)$ and solve the initial value problem with $y(a)$ and $y'(a)$.



We end up with $\overline{y}(b) < y(b)$, we undershot.

the shooting method, try again

We take $\overline{y}'(a)$ as a guess for $y'(a)$ and solve the initial value problem with $y(a)$ and $y'(a)$.



We end up with $\overline{y}(b) > y(b)$, we overshot.

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interpolation

- We used two different values for $\overline{y}'(a)$, let us call them Y_1 and Y_2.
- We obtained two different values for $y(b)$: say F_1 and F_2.

View F_1 as a function of Y_1 and F_2 as a function of Y_2:

$$F(Y_1) = F_1 \quad \text{and} \quad F(Y_2) = F_2.$$

We interpolate (remember Lagrange):

$$p(Y) = \left(\frac{Y_2 - Y}{Y_2 - Y_1} \right) F_1 + \left(\frac{Y_1 - Y}{Y_1 - Y_2} \right) F_2.$$

Observe: $p(Y_1) = F_1$ and $p(Y_2) = F_2$.

get ready for the next shot

To solve

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right), \quad x \in [a, b], \quad y(a) = y_a, \quad y(b) = y_b,$$

as an IVP, we tried Y_1 and Y_2 as values for $y'(a)$
and ended up with F_1 and F_2 for y_b, as written by the polynomial

$$p(Y) = \left(\frac{Y_2 - Y}{Y_2 - Y_1}\right) F_1 + \left(\frac{Y_1 - Y}{Y_1 - Y_2}\right) F_2.$$

For the next shot for $y'(a)$, we compute Y_3 via

$$p(Y_3) = u(b) \quad \text{and solve} \quad \left(\frac{Y_2 - Y_3}{Y_2 - Y_1}\right) F_1 + \left(\frac{Y_1 - Y_3}{Y_1 - Y_2}\right) F_2 = u(b).$$

Shooting Methods

linear ordinary differential equations

Consider the ordinary differential equation

$$y'' + Fy' + Gy = H,$$

with constant coefficients F, G, and H.

If y_1 and y_2 are solutions, then also $\dfrac{c_1 y_1 + c_2 y_2}{c_1 + c_2}$ is a solution for any two constants c_1 and c_2.

Let us verify:

$$\frac{c_1 y_1'' + c_2 y_2''}{c_1 + c_2} + F \frac{c_1 y_1' + c_2 y_2'}{c_1 + c_2} + G \frac{c_1 y_1 + c_2 y_2}{c_1 + c_2}$$

$$= \frac{c_1 (y_1'' + F y_1' + G y_1) + c_2 (y_2'' + F y_2' + G y_2)}{c_1 + c_2} = \frac{c_1 H + c_2 H}{c_1 + c_2} = H.$$

the falling object problem again

The falling object problem

$$y''(t) = \frac{d^2 y}{d\, t^2} = -g, \quad g = 9.81, \quad y(0) = 30, \quad y(4) = 0$$

is a linear problem, with exact solution $-g\dfrac{t^2}{2} + 12.12t + 30$.

Instead of a numerical solver, we can use the exact solution for when we have to compute $y(4)$ given a value for $y'(0)$.

Exercise 1:
Apply the shooting method to the falling object problem above, use $Y_1 = 10$ and $Y_2 = 14$ for the values for $y'(0)$.

You may use the exact solution instead of a numerical solver.

using `SciPy.integrate.RK45`

Exercise 2:
Consider the boundary value problem:

$$y''(t) = \frac{d^2 y}{d t^2} = -g, \quad g = 9.81, \quad y(0) = 30, \quad y(4) = 0.$$

Instead of using the exact solution of Exercise 1,
apply `RK45` of the `integrate` module of `SciPy` in the shooting method, using $Y_1 = 10$ and $Y_2 = 14$ for the values of $y'(0)$.

Shooting Methods

Dirichlet and Neumann conditions

Consider the boundary value problem

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right), \quad x \in [a, b].$$

We distinguish between two "pure" types of boundary conditions:

1. Dirichlet imposes conditions on values at end points.

$$y(a) = A \quad \text{and} \quad y(b) = B.$$

2. Neumann imposes conditions on derivatives at the ends.

$$y'(a) = A \quad \text{and} \quad y'(b) = B.$$

Conditions of "mixed" type combine values and derivatives.

the shooting method for Neumann conditions

We take $\overline{y}(a)$ as a guess for $y(a)$ and solve the initial value problem with $\overline{y}(a)$ and $y'(a)$.

1. Use two different values Y_1 and Y_2 for $\overline{y}(a)$ and label the corresponding values for $y'(b)$ as F_1 and F_2.
2. Let $p(Y)$ be the interpolating polynomial:

$$p(Y_1) = F_1 \quad \text{and} \quad p(Y_2) = F_2.$$

3. Determine Y_3 as the solution of $p(Y_3) = y'(b)$.

application of shooting to mixed conditions

Let $y(1) = 1$ and $y'(3) = 10.0179$ be two boundary conditions.

Suppose we solved two initial value problems:

- Taking $Y_1 = 1$ for $y(1)$ leads to $F_1 = 8.04819$ for $y'(3)$.
- Taking $Y_2 = 2$ for $y(1)$ leads to $F_3 = 11.6751$ for $y'(3)$.

Since $y'(3) = 10.0179$, with $Y_1 = 1$ we undershot the solution, and with $Y_2 = 2$ we overshot the solution.

Let us predict the next value for $y(1)$ by linear interpolation:

$$\begin{aligned} p(Y) &= \left(\frac{Y-1}{2-1} \right) 11.6751 + + \left(\frac{Y-2}{1-2} \right) 8.04819 \\ &= 3.62691\, Y + 4.42128. \end{aligned}$$

interpolation continued

$$
\begin{aligned}
p(Y) &= \left(\frac{Y-1}{2-1}\right) 11.6751 + \left(\frac{Y-2}{1-2}\right) 8.04819 \\
&= 3.62691\, Y + 4.42128.
\end{aligned}
$$

We have to find Y_3 such that $p(Y_3) = y'(3) = 10.0179$:

$$
Y_3 = \frac{10.0179 - 4.42128}{3.62619} = 1.54308.
$$

The next IVP will be solved with 1.54308 as the value for $y(1)$.

an example

Consider $y'' = y$, with $y'(1) = 1.17520$ and $y'(3) = 10.0179$.

- $y(1) = 1$ leads to $y'(3) = 8.04819$
- $y(1) = 2$ leads to $y'(3) = 11.6751$

The equation $y'' = y$ is linear, which means that

for any two solutions y_1, y_2, also $\dfrac{c_1 y_1 + c_2 y_2}{c_1 + c_2}$ is a solution.

$$y(x) = \frac{c_1 y_1 + c_2 y_2}{c_1 + c_2} \quad \Rightarrow \quad y'(x) = \frac{c_1}{c_1 + c_2} y_1' + \frac{c_2}{c_1 + c_2} y_2'.$$

Let $\gamma_1 = c_1/(c_1 + c_2)$ and and $\gamma_2 = c_2/(c_1 + c_2)$, then:

$$\begin{array}{lclcl} y'(1) & = & 1.17520 & = & \gamma_1 y_1'(1) + \gamma_2 y_2'(1) \\ y'(3) & = & 10.0179 & = & \gamma_1 y_1'(3) + \gamma_2 y_2'(3) \end{array}$$

a linear system

$$\begin{array}{rcccl} y'(1) & = & 1.17520 & = & \gamma_1 y_1'(1) + \gamma_2 y_2'(1) \\ y'(3) & = & 10.0179 & = & \gamma_1 y_1'(3) + \gamma_2 y_2'(3) \end{array}$$

We have two computed solutions: $y_1'(3) = 8.04819$ and $y_2'(3) = 11.6751$. Moreover: $y_1'(1) = 1.17520 = y_2'(1)$.

Thus we can solve the system for γ_1 and γ_2:

$$\left[\begin{array}{cc} 1.17520 & 1.17520 \\ 8.04819 & 11.6751 \end{array} \right] \left[\begin{array}{c} \gamma_1 \\ \gamma_2 \end{array} \right] = \left[\begin{array}{c} 1.17502 \\ 10.0179 \end{array} \right]$$

This example shows that for linear problems, solving two initial value problems suffices to find the solution to the boundary value problem.

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a nonlinear boundary value problem

Consider

$$\frac{d^2y}{dx^2} - \left(1 - \frac{x}{5}\right) y(x) \frac{dy}{dx} = x, \quad x \in [1, 3],$$

with conditions on the boundaries

$$y(1) = 2 \quad \text{and} \quad y(3) = -1.$$

Observe that the coefficients depend on x.

This is a *nonlinear* boundary value problem.

Linear problems can be solved with linear interpolation.
For nonlinear problems, higher degree interpolation is needed.

solving a nonlinear boundary value problem

Consider

$$\frac{d^2y}{dx^2} - \left(1 - \frac{x}{5}\right) y(x) \frac{dy}{dx} = x, \quad x \in [1, 3],$$

with conditions on the boundaries

$$y(1) = 2 \quad \text{and} \quad y(3) = -1.$$

We solve this problem by shooting,
with a reduction to an initial value problem.

We know $y(1) = 2$, but what is $y'(1)$?

$$p(y'(1)) = y(3) = -1$$

We predict the value for $y'(1)$ to reach -1.

a third exercise

Exercise 3:

Consider the boundary value problem

$$y''(t) + t\,y'(t) - y(t) = t^2, \quad y(0) = 0, \quad y(5) = 20.$$

Using 1 and 2 as guesses for $y'(0)$
gave respectively 19.47 and 24.47 for $y(5)$.

1. What is your next guess Y_3 in the shooting method?
2. Use `SciPy.integrate.RK45` to compute the value for $y(5)$ when solving the initial value problem with $y(0) = Y_3$.

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the pendulum as a nonlinear BVP

A model of a pendulum, without damping, without forcing:

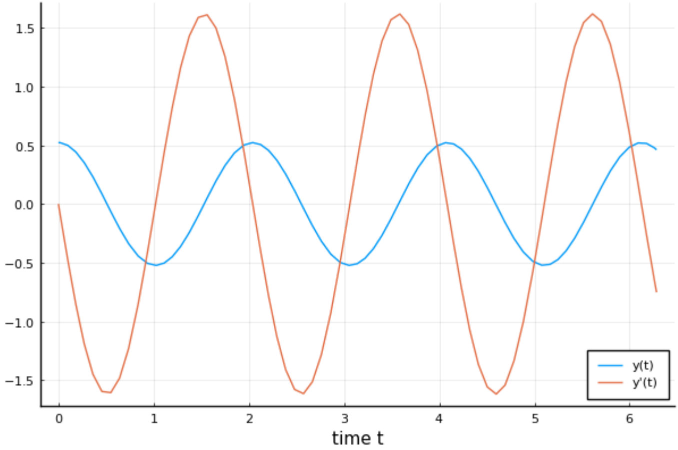
$$\frac{d^2 y}{dt^2} = -g \sin(y),$$

where g is the constant approximately equal to 9.807.

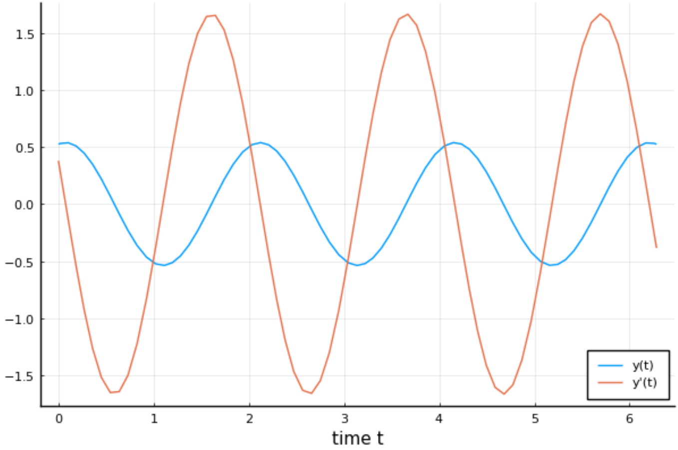
Formulated as a Boundary Value Problem, we add two conditions:

$$y(0) = \pi/6, \quad y(2\pi) = \pi/6.$$

with initial velocity $y'(0) = 0$



the third trajectory



shooting with linear interpolation

After $y'(0) = 0$, the second initial velocity is $y'(0) = 1$.

The end angles are the values of the interpolating polynomial p:

$$\begin{aligned} p(0) &= 0.463 \\ p(1) &= 0.605 \end{aligned}$$

The interpolating polynomial is

$$p(z) = \left(\frac{z-1}{0-1} \right) 0.463 + \left(\frac{z-0}{1-0} \right) 0.605.$$

Solving $p(z) - \pi/6 = 0$ gives $z = 0.428$.

shooting with quadratic interpolation

After $y'(0) = 0$, the second initial velocity is $y'(0) = 1$.

The end angles are the values of the interpolating polynomial q:

$$q(0) = 0.463, \quad q(1) = 0.605, \quad q(0.428) = 0.531.$$

The interpolating polynomial is

$$\begin{aligned} q(z) &= \left(\frac{z-1}{0-1}\right)\left(\frac{z-0.428}{0-0.428}\right)0.463 \\ &+ \left(\frac{z-0}{1-0}\right)\left(\frac{z-0.428}{1-0.428}\right)0.605 \\ &+ \left(\frac{z-0}{0.428-0}\right)\left(\frac{z-1}{0.428-1}\right)0.531. \end{aligned}$$

Solving $q(z) - \pi/6 = 0$ gives $z = 0.379$ and $z = 5.37$.
Take the closest solution to the previous initial velocity: $z = 0.379$.

the pendulum with damping and forcing

The extended model of the pendulum adds damping and forcing:

$$\frac{d^2y}{dt^2} + A\frac{dy}{dt} + g\sin(y) = B\sin(t), \quad A = 1/10, \quad B = 10,$$

with boundary values $y(0) = \pi/6$, $y(2\pi) = \pi/6$.

Exercise 4: Apply the shooting method to this problem in two ways:

1. With linear extrapolation, using the last two guesses for $y'(0)$.
2. With quadratic extrapolation, using the last three guesses.

Organize your computations in a systematic manner.

Do you observe a difference between linear and quadratic extrapolation in the number of shooting steps needed to obtain correct values for $y'(0)$?