

## MCS 471: Review of Chapters 0, 1, and 2

The questions below are meant to stimulate the preparation of the final exam, concerning chapters 0, 1, and 2. They cover some of the most important but not all topics. You must review all course materials: textbook, homework exercises, lecture notes (also consult the course web site), and quizzes. Also please review the computer projects.

1. Consider the representation of floating-point numbers with base 2 and 3 digits in the fraction part. The values for the exponents are between  $-16$  and  $+16$ .
  - (a) What is the machine precision in this number system?
  - (b) Represent the numbers 7 and 6 as floating point numbers and illustrate the calculation of  $7 + 6$  and  $7 - 6$ .
2. Let  $f(x) = \frac{x}{x-1}$ . Consider the following values for  $x$ : a random number, a point close to the origin (i.e.:  $|x| < \delta$ ), and a point close to 1 (i.e.:  $|x - 1| < \delta$ ). Answer the following:
  - (a) What is the difference in relative and absolute error in the evaluation of  $f(x)$  for these three points?
  - (b) Assuming there is a small error of  $\delta$  on the three points, what is effect of this error on the function value?
  - (c) Discuss the difference in numerical conditioning between the root finding and the evaluation problem.
3. Find an approximation for the square root of 4.12.
  - (a) Apply three steps of the bisection method, starting with the interval  $[2, 2.1]$ . Complete the following table, using four decimal places:

$k$	$a_k$	$b_k$	$\frac{a_k + b_k}{2}$	$f(a_k)$	$f(b_k)$	$f(\frac{a_k + b_k}{2})$
0	2.000	2.100				
1						
2						

- (b) Apply three steps of the secant method, starting with  $x_0 = 1$  and  $x_1 = 2$ . Complete the following table, using four decimal places:

$k$	$x_k$	$x_{k+1}$	$x_{k+2}$	$f(x_k)$	$f(x_{k+1})$	$f(x_{k+2})$
0	1.000	2.000				
1						
2						

- (c) Use Newton's method, starting at 2, until you are sure you have eight decimal places correct. Make a table with four columns, containing the respective values for  $k$ ,  $x_k$ ,  $f(x_k)$ , and  $f(x_k)/f'(x_k)$ . Write  $x_k$  with as many decimal places as your calculator shows. Give the values for  $f(x_k)$  and  $f(x_k)/f'(x_k)$  in scientific format, using four decimal places.
4. Derive the formula for the secant method to find an approximation for the root of  $f(x) = 0$ .
5. Apply three steps of the golden section search method to find then minimum of  $f(x) = x^3 - x$  in the interval  $[0, 1]$ .  
Write the values for  $a$ ,  $b$ ,  $x_1$ ,  $x_2$ ,  $f(x_1)$ , and  $f(x_2)$  in the table (4 decimal places):

step	$a$	$b$	$x_1$	$x_2$	$f(x_1)$	$f(x_2)$
0	0.000	1.000				
1						
2						

6. Using the formulas for  $x_1$  and  $x_2$ , show that the golden section search method always reduces the search interval by a factor of  $c$ .
7. What is error propagation? Illustrate the propagation of errors with a method we have seen during the course.
8. The following fixed-point iterations all have the same fixed point, i.e.: 1.
- $x_{k+1} = 3 - 2x_k^3$ ,  $k = 0, 1, \dots$
  - $x_{k+1} = \frac{3}{2x_k^2 + x_k}$ ,  $k = 0, 1, \dots$
  - $x_{k+1} = \frac{4x_k^3 + 3}{6x_k^2 + 1}$ ,  $k = 0, 1, \dots$

For each of the three fixed-point iterations, make a cobweb picture illustrating the convergence (or divergence), starting at  $x_0 = 0.9$ . Compute the convergence (or divergence) rate for each iteration. Which iteration is best?

9. Show that Newton's method converges linearly with ratio  $\frac{m-1}{m}$  to a root of multiplicity  $m$ .
10. Consider the linear system

$$\begin{cases} -1.000x_1 - 1.000x_2 + 1.000x_3 = -1.000 \\ 1.000x_1 - 1.000x_2 + 0.000x_3 = 0.000 \\ 2.000x_1 + 0.000x_2 + 1.000x_3 = 3.000 \end{cases}$$

- Use Gaussian elimination to compute an LU decomposition in the following two ways:
  - without partial pivoting;
  - with partial pivoting;
- Solve the system two times, using the two LU decompositions obtained from above.
- Compute the determinant using the two LU decompositions obtained from above.
- Use the second LU decomposition you obtained to compute  $A^{-1}$  and  $\text{cond}(A)$  with  $\|\cdot\|_1$ .

**FINAL EXAM is in BH 0308 on Monday 8 December 2003, from 1 till 3PM.**