# MCS 471 Project Five: Solving ODEs with MATLAB

The goal of this project is to study numerical solutions to Ordinary Differential Equations (ODEs). We use MATLAB, version 6.1.

## Some useful commands in MATLAB

The command **ode45** is a pair of Runge-Kutta methods, of orders four and five. To solve the test problem

$$\frac{dy}{dx} = y, \quad y(0) = 1,$$

numerically, we fix a range for $x$, say the interval $[0, 1]$, and define the ODE then as follows:

```
>> xspan = [0 1];                          % x in interval [0,1]
>> yzero = 1;                              % initial value y(0) = 1
>> f = inline('y','x','y');                % define f(x,y) = y, as function of x and y
```

The inline is convenient to define a "one line" function, from a string. In the command above we define the function $f(x, y)$ as $y$. Then we call **ode45** as follows:

```
>> [x y] = ode45(f,xspan,yzero);
```

We could have left out the ';' after each command, but here is the really interesting output:

```
>> plot(x,y)                               % plot the solution
>> max(abs(y-exp(x)))                      % compare to the exact solution
```

To examine the sensitivity of the computed solution with regard to changes in the initial values, we conduct the following experiment:

```
>> [x0 y0] = ode45(f,[0 3],yzero);         % solve over interval [0,3]
>> [x1 y1] = ode45(f,[0 3],yzero+0.1);     % perturb initial value
>> plot(x0,y0); hold on; plot(x1,y1);      % plot solutions on same figure
>> abs(y0(end)-y1(end))                    % error at the end of [0,3]
```

As we see, the solutions grow apart, and will keep growing apart if we take larger and larger values for $x$, so the initial value problem is ill conditioned.

A simple model of a pendulum is given by the following second-order equation:

$$\frac{d^2}{dt^2}\theta(t) + \sin(\theta(t)) = 0.$$

To solve it, we first must rewrite the second-order equation as a system of two first-order equations:

$$\begin{cases} \frac{d}{dt}y_1(t) = y_2(t) \\ \frac{d}{dt}y_2(t) = -\sin(y_1(t)) \end{cases} \quad \text{where} \quad y_1 = \theta.$$

```
>> f = inline('[y(2); -sin(y(1))]','t','y'); % define right-hand side f
>> yzero = [1 0];                          % initial values y(1) = 1, y(2) = 0
>> tspan = [0 10];                         % t in interval [0,10]
>> [t y] = ode45(f,tspan,yzero);           % solve the ODE
>> plot(t,y(:,1))                          % plot the values for theta
```

**Assignment One.** Consider the initial value problem

$$\frac{dy}{dx} = \frac{1}{1 - 0.2\cos(y)}, \quad y(0) = 0, \quad x \in [0, 2\pi].$$

1. Verify that the exact solution $y = y(x)$ satisfies Kepler's equation $y - 0.2\sin(y) - x = 0$. Solve this ODE in MATLAB with exact initial value $y(0) = 0$. What is the largest number (in absolute value) when you substitute the computed values for $y$ in Kepler's equation?
2. Examine the sensitivity of the solution to changes in the initial condition, solving the initial value problem for $y(0) = \delta$, for $\delta \in \{1, 0.1, 0.01, 0.001\}$. Give the four plots of the solutions $y_\delta(t)$. Make a table with the errors $|y_\delta(2\pi) - y(2\pi)|$. Is this problem well conditioned?

**Assignment Two.** A more realistic model of a pendulum considers air resistance:

$$\frac{d^2}{dt^2}\theta(t) + k\frac{d}{dt}\theta(t) + \sin(\theta(t)) = 0, \quad \theta(0) = 1, \quad \frac{d}{dt}\theta(0) = 0.$$

With a proper choice of $k$, the pendulum comes to a rest position.

1. Solve this problem with $k = 0.2$. Choose the interval for $t$ large enough so that the amplitude for $\theta(t)$ in the end becomes smaller than 0.01. Plot the solution for $\theta(t)$. How long does it take till the pendulum has an amplitude smaller than 0.01?
2. Solve this problem with $k = -0.04$. Choose the interval for $t$ large enough so that the amplitude for $\theta(t)$ in the end becomes larger than 20. Plot the solution for $\theta(t)$. How long does it take till the pendulum has an amplitude larger than 20?
3. Compare the solutions for $k = 0.2$ and $k = -0.04$. Solve the problems taking $\theta(0) = 1.01$ instead of $\theta(0) = 1$, to examine the sensitivity of the solution with respect to changes to $\theta(0)$. For which values of $k$ is the initial value problem well conditioned? Interpret your answer, referring to the plots.

**Assignment Three.** Consider the following example (formula (6.29) in textbook page 484):

$$\left\{ \begin{array}{lll} \frac{d}{dt}y_1(t) = 1195y_1(t) - 1995y_2(t) & y_1(0) = 2 & y_1(t) = 10e^{-2t} - 8e^{-800t} \\ \frac{d}{dt}y_2(t) = 1197y_1(t) - 1997y_2(t) & y_2(0) = -2 & y_2(t) = 6e^{-2t} - 8e^{-800t}. \end{array} \right.$$

1. Solve this problem using **ode45** for $t \in [0, 1]$. What is the largest error between the exact and computed approximation? How many steps have been used?
2. For this *stiff* problem, it is better to use **ode15s**, so do this for $t \in [0, 1]$. What is the largest error between the exact and computed approximation? How many steps have been used?
3. Compare the results obtained from **ode45** and **ode15s**.

**The deadline is Wednesday 3 December, at 1PM.** Bring your project solution to class. It should contain the following:

1. Tables with numerical data, numbers formatted in scientific format, and the plots.
2. Answers to the questions in the assignments. Please write complete grammatically correct sentences and avoid spelling mistakes.
3. Sequences of commands used to generate the data for the experiments. It is good practice to store these commands in little .m files, e.g., in `assignment_one.m`, etc.
4. Output of your sessions with MATLAB as *an appendix*. Typing **diary** followed by the name of a file in a session creates a new file with the given name which will contain everything you see on the screen during the session. You may edit out mistakes from the output of diary or truncate. This output is only as a backup, used for partial credit (if needed).

If you have questions, comments, or difficulties, feel free to come to my office for help.