MCS 471 Project Five: Solving ODEs with MATLAB

The goal of this project is to study numerical solutions to Ordinary Differential Equations (ODEs). We use MATLAB, version 6.1, or higher.

Some useful commands in MATLAB

The command **ode45** is a pair of Runge-Kutta methods, of orders four and five. To solve the test problem

$$\frac{dy}{dx} = y, \quad y(0) = 1,$$

numerically, we fix a range for x, say the interval $[0, 1]$, and define the ODE then as follows:

```
>> xspan = [0 1];                        % x in interval [0,1]
>> yzero = 1;                            % initial value y(0) = 1
>> f = inline('y','x','y');              % define f(x,y) = y, as function of x and y
```

The inline is convenient to define a "one line" function, from a string. In the command above we define the function $f(x, y)$ as y. Then we call **ode45** as follows:

```
>> [x y] = ode45(f,xspan,yzero);
```

We could have left out the ';' after each command, but here is the really interesting output:

```
>> plot(x,y)                             % plot the solution
>> max(abs(y-exp(x)))                    % compare to the exact solution
```

To examine the sensitivity of the computed solution with regard to changes in the initial values, we conduct the following experiment:

```
>> [x0 y0] = ode45(f,[0 3],yzero);       % solve over interval [0,3]
>> [x1 y1] = ode45(f,[0 3],yzero+0.1);   % perturb initial value
>> plot(x0,y0); hold on; plot(x1,y1);    % plot solutions on same figure
>> abs(y0(end)-y1(end))                  % error at the end of [0,3]
```

As we see, the solutions grow apart, and will keep growing apart if we take larger and larger values for x, so the initial value problem is ill conditioned.

A simple model of a pendulum is given by the following second-order equation:

$$\frac{d^2}{dt^2}\theta(t) + \sin(\theta(t)) = 0.$$

To solve it, we first must rewrite the second-order equation as a system of two first-order equations:

$$\begin{cases} \frac{d}{dt}y_1(t) = y_2(t) \\ \frac{d}{dt}y_2(t) = -\sin(y_1(t)) \end{cases} \quad \text{where} \quad y_1 = \theta.$$

```
>> f = inline('[y(2); -sin(y(1))]','t','y'); % define right-hand side f
>> yzero = [1 0];                        % initial values y(1) = 1, y(2) = 0
>> tspan = [0 10];                       % t in interval [0,10]
>> [t y] = ode45(f,tspan,yzero);         % solve the ODE
>> plot(t,y(:,1))                        % plot the values for theta
```

Assignment One. Consider the initial value problem

$$\frac{dy}{dx} = \frac{1}{1 - 0.2\cos(y)}, \quad y(0) = 0, \quad x \in [0, 2\pi].$$

The commands **ode23** and **ode45** implement pairs of Runge-Kutta-Fehlberg methods with a variable step size, respectively of orders 2 and 4, using methods of orders 3 and 5 to estimate the errors.

1. Use **ode23** and **ode45** to solve this problem. The exact solution $y = y(x)$ satisfies Kepler's equation $y - 0.2\sin(y) - x = 0$. Kepler's equation defines the residual of the solutions obtained by **ode23** and **ode45**. Use the `norm` command to compare the accuracy of both methods.
2. How many steps did **ode23** need to compute $y(2\pi)$?
   Compare this number with the number of steps **ode45** needed to reach $x = 2\pi$.

Assignment Two. The ejection of a cork from a bottle containing a fermenting liquid is modeled by

$$\frac{d^2x}{dt^2} = g(1+q)\left(\left(1 + \frac{x}{D}\right)^{-\gamma} + \frac{Rt}{100} - 1 + \frac{qx}{L(1+q)}\right),$$

depending on five parameters: $g = 9.81$, $q = 20$, $D = 5$, $\gamma = 1.4$, and $R = 4$. This ODE describes the evolution of the displacement $x(t)$ of the cork at time t. As the length of the cork we take $L = 3.75$. For $x(t) \leq L$, the cork is still in the neck of the bottle.

The "event location problem" is to determine the moment t^* when $x(t^*) = L$, i.e.: when the cork leaves the bottle. Equally interesting is the value of $x'(t^*)$, i.e.: the speed of the cork at t^*.

1. Solve the event location problem for initial conditions: $x(0) = 0$ and $x'(0) = 0$, when the cork starts at rest. Determine t^*, with an accuracy satisfying: $0 \leq x(t^*) - L < 5\ 10^{-3}$.
2. Use the value of $x'(t^*)$ to describe the sensitivity of t^* to changes in the length L.
   Interpret your answer referring to the plot of $x(t)$ around t^*.
3. Solve the event location problem for several values of the initial conditions to determine whether the solution is more sensitive to changes in $x(0)$ than in $x'(0)$, or vice versa. Fix $x(0) = 0$ and determine t^* for small values of $x'(0)$. Repeat for small values of $x(0)$, fixing $x'(0) = 0$. Illustrate your answer by referring to the plots of $x(t)$ and $x'(t)$ for small values of t.

Assignment Three. Consider the following system of first order differential equations:

$$\begin{cases} \frac{d}{dt}y_1(t) = 0.08\ y_1(t) + 1.44\ y_2(t) \\ \frac{d}{dt}y_2(t) = 1.44\ y_1(t) + 0.92\ y_2(t) \end{cases} \quad y_1(0) = -4, y_2(0) = 3.$$

1. Solve this problem using **ode45** for $t \in [0, 10]$. Plot the solution $(y_1(t), y_2(t))$.
2. Change the initial values slightly, e.g.: $y_1(0) = -4.001$ and $y_2(0) = 3.001$. Solve the problem again and describe the solution trajectory. What does this imply about the sensitivity of the solution $(y_1(t), y_2(t))$ to changes in the initial values $(y_1(0), y_2(0))$?
3. Use **eig** to compute the eigenvalues and eigenvalues of the matrix $A = \begin{bmatrix} 0.08 & 1.44 \\ 1.44 & 0.92 \end{bmatrix}$.
   Interpret the computational experiments you did, using the eigenvalues and eigenvectors of A.

The deadline is Friday 2 December, at 3PM. Your project solution should contain the following:
(1) Tables with numerical data, numbers formatted in scientific format, and the plots;
(2) Answers to the questions asked (please write complete and grammatically correct sentences);
(3) Sequences of commands used to generate the data for the experiments;
(4) Output of your sessions with MATLAB as *an appendix* (use diary).

If you have questions, comments, or difficulties, feel free to come to my office for help.