MCS 471 Project One: Numerical Evaluation of Quadratic Form

The goal of the project is to investigate numerical aspects of evaluating quadratic forms. We will use Maple in our investigations. Maple 9.5 is available in the computer labs on campus. To start working on the project, download the companion Maple worksheet from the web pages for this course.

0. The Numerical Evaluation of Quadric Forms

A quadratic form in n variables is defined by an n-by-n symmetric matrix A, an n-vector b, and a constant c:

$$p(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} + b^T\mathbf{x} + c, \quad \mathbf{x}^T = [x_1 \;\; x_2 \;\; \cdots \;\; x_n]. \tag{1}$$

By default, vectors are column vectors, so we use the transpose ^{T} to turn a column vectors $\mathbf{x}$ and b into a row vectors $\mathbf{x}^T$ and b^T.

We use the `LinearAlgebra` package in Maple, our default precision is 8 decimal places, and the seed to generate (pseudo-)random numbers is the number on your student ID card:

```
[> with(LinearAlgebra):
[> Digits := 8:              # working precision
[> randomize(872982372323): # use your university identity number
```

0.1. Approximate and Exact Data

We start out generating random numbers to define (A, b, c).

```
[> n := 5:  # the number of variables
[> A := RandomMatrix(n,n,generator=-1.0..1.0,outputoptions=[shape=symmetric]):
[> exact_A := convert(A,rational):
[> b := RandomMatrix(n,1,generator=-1.0..1.0):
[> exact_b := convert(b,rational):
[> c := stats[random,uniform[-1.0,1.0]](1):
[> exact_c := convert(c,rational):
[> v := seq(x[i],i=1..n):  # sequence of variable names
[> X := Vector(n,[v]):     # vector of variable names, X is capitalized
[> p := expand(Transpose(X).A.X) + (Transpose(b).X)[1] + c;
```

The result of the last command is a quadratic polynomial in 5 variables, with approximate coefficients. The exact polynomial is calculated using the exact data, our representation error is of order 10^{-8}, since our working precision is 8 decimal places.

```
[> exact_p := expand(Transpose(X).exact_A.X) + (Transpose(exact_b).X)[1] + exact_c;
[> representation_error := exact_p - p;
```

0.2. Functions to Evaluate Quadrics

We can evaluate the quadric as a fully expanded polynomial, or in its matrix form. We use a random point to test our evaluation procedures:

```
[> r := stats[random,uniform[-1.0,1.0]](n):
[> exact_r := op(convert([r],rational)):
[> R := Vector(n,[r]):
[> exact_R := convert(R,rational):
```

The little `r` is a sequence of numbers and will be used to evaluate the quadratic form as a polynomial, using the `f1` (and its exact counterpart `exact_f1`) below. The big `R` is a vector and is input to the `f2` (and `exact_f2`) which evaluates the quadratic form using matrix-vector operations. The following Maple commands define the functions to evaluate the quadratic form, respectively by `f1` and `f2` as an expanded polynomial and in its more compact matrix-vector format.

```
[> f1 := unapply(p,v);
[> exact_f1 := unapply(exact_p,v);
[> f2 := X -> Transpose(X).A.X + (Transpose(b).X)[1] + c;
[> exact_f2 := X -> Transpose(X).exact_A.X + (Transpose(exact_b).X)[1] + exact_c;
```

To evaluate, we type:

```
[> f1(r); exact_f1(exact_r);
[> f2(R); exact_f2(exact_R);
```

The results of `exact_f1` and `exact_f2` will always agree, but there is a difference between the outcomes of `f1` and `f2`.

1. Which way to evaluate a quadric is most efficient?

We count the cost of a function by adding the number of arithmetical operations (additions, subtractions, and multiplications) it takes to compute the value of the function. We consider a subtraction as having the same cost as an addition.

Assignment One. Give high level algorithms for the functions `f1` and `f2`, using pseudo-code making the additions and multiplications explicit. Count the number of arithmetical operations it takes to evaluate `f1` and `f2`. Justify your count referring to the algorithmic descriptions of `f1` and `f2`. Start with $n = 3$, and continue with $n = 4$, $n = 5$, etc, till you find a general formula in terms of n. Which one is most efficient, `f1` or `f2`?

2. Which way to evaluate a quadric is most accurate?

We consider three types of points: random points, points of large magnitude, and the center of the quadric. The question: which one is most accurate, `f1` or `f2`, has to be answered in three parts.

2.1. Evaluation at Random Points

A random point is generated first as a sequence of random numbers (input to `f1`), then converted into a sequence rational numbers (input to `exact_f1`) and into a vector of numbers (input to `f2`).

```
[> r := stats[random,uniform[-1.0,1.0]](n);
[> exact_r := op(convert([r],rational));
[> R := Vector(n,[r]);
[> y1 := f1(r); y2 := f2(R); y0 := exact_f1(exact_r);
```

To compute the relative errors, we temporarily increase the working precision:

```
[> Digits := 16;
[> relative_error1 := abs(y1-y0)/abs(y0); relative_error2 := abs(y2-y0)/abs(y0);
[> Digits := 8;
```

Assignment Two – part one. Generate 10 random points, evaluate at `f1`, `f2`, and compute the relative error using the exact value. Record the magnitude of these relative errors in a table for $n = 5, 10, 15, 20$. Write down in a couple of sentences what you observe from the table of errors. In particular: is there any variation in the magnitudes of errors for the same n, and as n increases? Is there a clear difference in accuracy between `f1` and `f2`?

2.2. Evaluation at Points of Large Magnitude

In this part we keep the n fixed to 5, but let the magnitudes of the points decrease or increase. For example, to get a random point of magnitude 10^4, we compute

```
[> r4 := 10^4*r: exact_r4 := op(convert([r4],rational)): R4 := Vector(n,[r4]):
[> y1 := f1(r4); y2 := f2(R4); y0 := exact_f1(exact_r4);
```

Also here we raise the precision to compute the errors:

```
[> Digits := 16:
[> absolute_error1 := abs(y1-y0); absolute_error2 := abs(y2-y0);
[> relative_error1 := absolute_error1/abs(y0); relative_error2 := absolute_error2/abs(y0);
[> Digits := 8:
```

Assignment Two – part two. Generate points of magnitudes 10^{-8}, 10^{-6}, 10^{-4}, 10^{-2}, 10^2, 10^4, 10^6, 10^8. Evaluate `f1`, `f2`, and compute the absolute and relative error using the exact value. For these eight points, write down the absolute and relative errors in a table, using scientific notation of the errors. Do you see any relation between the errors and the magnitudes of the random points? Is there a clear difference between `f1` and `f2`? Write a couple of sentences to summarize your experiments.

2.3. Evaluation at a Special Point

The special point of interest is the so-called center of the quadric, defined as the solution to $A\mathbf{x} = -b$. Evaluating at the center should give us c back. We first calculate the center exactly before rounding it to our working precision.

```
[> exact_S := Column(LinearSolve(exact_A,-exact_b),1):
[> exact_s := op(convert(exact_S,list)):
[> s := evalf(exact_s):
[> S := Vector(n,[s]):
[> f1(s); f2(S);
[> exact_f1(exact_s) = exact_c;
```

Assignment Two – part three. In this assignment, we fix n to be 5, but let c decrease. Consider values of c of magnitude 10^{-2}, 10^{-4}, 10^{-6}, 10^{-8}, and 10^{-10}. Evaluate with `f1`, `f2`, and compute the absolute and relative error, using the exact value for c. Record these errors (in scientific notation) in a table. Do you now see a difference between f1 and f2?

Assignment Two – conclusion. Based on all experiments, write a conclusion in one paragraph.

3. The deadline is Monday 12 September 2005 at 3PM

Bring *your* solution to the project to class. The *your* is emphasized to stress that your solution is the result of an *individual* effort. Collaborations are **not** permitted.
Your solution should contain the following:

1. Answers to the questions in the assignments. Please write complete grammatically correct sentences.
2. Tables summarizing the numerical experiments you have done.
3. A print out of the Maple worksheet to show the set up of your calculations may be included *as an appendix.* Suppress long output with colons.

Do not eMail me Maple worksheets. Also, summarize your calculations, it is not necessary to print out every single case you computed.

If you have questions or difficulties with the assignments, feel free to come to my office for help.