

# Numerical Conditioning

- 1 Forward and Backward Error
  - nearby solutions for nearby problems
- 2 Multiple Roots
  - where functions are very flat
- 3 The Wilkinson Polynomial
  - a historical example in numerical analysis
- 4 Condition Number
  - the error magnification factor

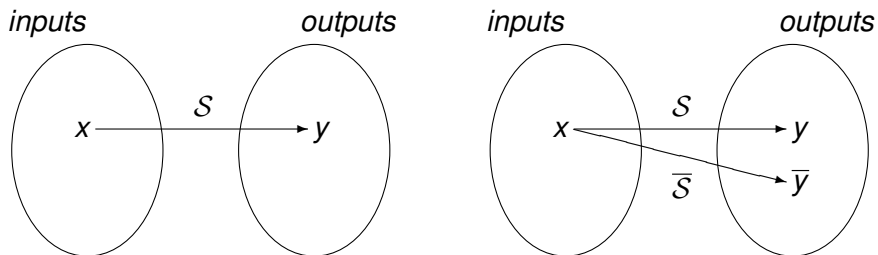
MCS 471 Lecture 4  
Numerical Analysis  
Jan Verschelde, 30 August 2021

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- 1 **Forward and Backward Error**
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## nearby solutions for nearby problems

Consider a solution map  $\mathcal{S}$  between the space of *inputs* and *outputs*.



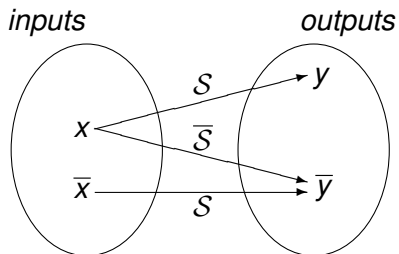
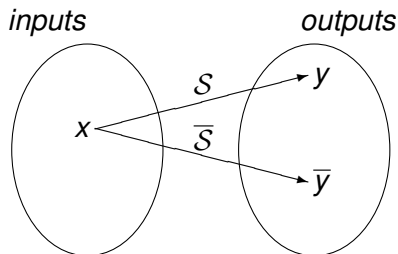
Because of floating-point arithmetic, we obtain  $\bar{y} = \bar{\mathcal{S}}(x)$ .

### Definition

If  $y = \mathcal{S}(x)$  is the exact solution for input  $x$  and  $\bar{y} = \bar{\mathcal{S}}(x)$  is the approximate solution for the input  $x$ , then **the forward error** is  $|y - \bar{y}|$ .

## forward error and backward error

For  $y = S(x)$  and  $\bar{y} = \bar{S}(x)$ , the forward error is  $|y - \bar{y}|$ .  
Modify the input  $x$  into  $\bar{x}$  so that  $\bar{y} = S(\bar{x})$ .



### Definition

If  $y = S(x)$  is the exact solution for input  $x$ ,  $\bar{y} = \bar{S}(x)$  is the approximate solution for the input  $x$ , and  $\bar{x}$  is the modified input so that  $\bar{y} = S(\bar{x})$ , then *the backward error* is  $|x - \bar{x}|$ .

## application to the root finding problem

Solve the equation  $f(x) = 0$  to find a root  $r$ .

We find an approximation  $\bar{r}$ :  $f(\bar{r}) \approx 0$ .

Denote  $\Delta r$  such that  $\Delta r = r - \bar{r}$  or  $r = \bar{r} + \Delta r$ .

We have  $f(r) = 0$  and thus  $f(\bar{r} + \Delta r) = 0$ ,  $|\Delta r|$  is the forward error.

Denote  $\Delta f$  such that  $\Delta f = f - \bar{f}$ , where  $\bar{f}(\bar{r}) = 0$ .

$$\begin{aligned}\bar{f}(\bar{r}) = 0 &\Leftrightarrow (f - \Delta f)(\bar{r}) = 0 \\ &\Leftrightarrow f(\bar{r}) = \Delta f(\bar{r})\end{aligned}$$

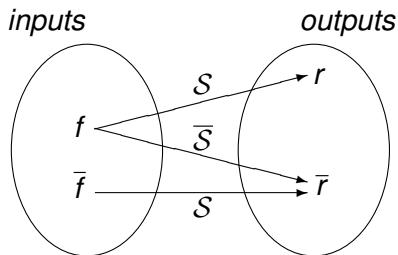
### Definition (backward error of the root finding problem)

Let  $r$  be a root of  $f$ . The *backward error* is  $|\Delta f(\bar{r})| = |f(\bar{r})|$ .

Although we do not know  $\Delta r$ ,  $f(\bar{r})$  is a simple evaluation.

## relation between forward and backward error?

For the root finding problem:  $r = \mathcal{S}(f)$ ,  $\bar{r} = \bar{\mathcal{S}}(f)$ , and  $\bar{r} = \mathcal{S}(\bar{f})$ .



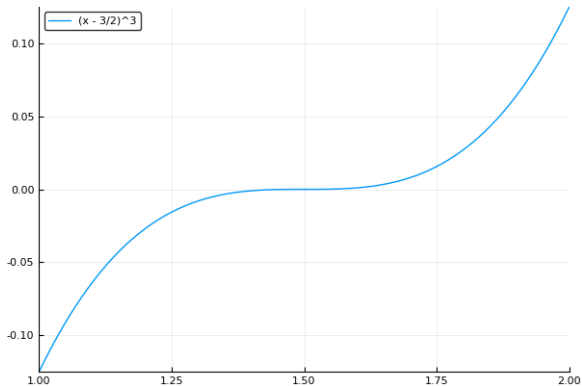
As the backward error  $|\Delta f| = |f(\bar{r})|$ , is  $|\Delta f| \approx |\Delta r|$ ?

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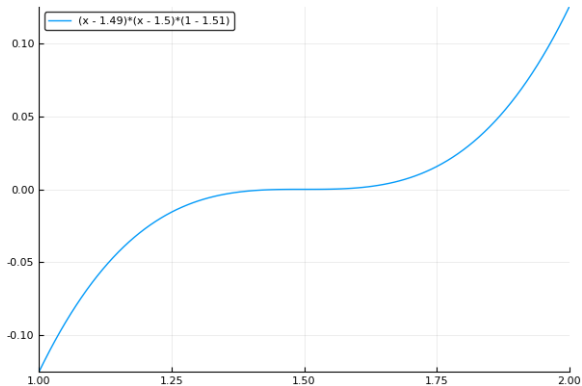
## a multiple root

Consider the plot of  $(x - 3/2)^3$ :



## close to a multiple root

Consider the plot of  $(x - 1.49)(x - 1.5)(x - 1.51)$ :



## relation between forward and backward error?

Consider  $p(x) = (x - 1.5)^3$  and  $q(x) = (x - 1.51)(x - 1.5)(x - 1.49)$ .  
The forward error is the error in the roots:  $|\Delta r| = 0.01 = 10^{-2}$ .

Consider the difference in the coefficients of  $p$  and  $q$ :

$$p(x) = x^3 - 4.5x^2 + 6.75x - 3.375$$

$$q(x) = x^3 - 4.5x^2 + 6.7499x - 3.37485$$

$|6.75 - 6.7499| = 10^{-4}$  and  $|3.375 - 3.37484| = 1.50 \cdot 10^{-4}$ .

Adding up the errors, we find  $|\Delta p| = 2.50 \cdot 10^{-4}$ .

Now we compare:  $|\Delta p| = 2.50 \cdot 10^{-4} \ll 10^{-2} = |\Delta r|$ .

*The magnitude of the error on the root (the output) is much larger than the magnitude of the error on the coefficients (the input).*

Observe: evaluate the coefficients of  $q$  to three decimal places ...

## sensitivity formula for roots

The equation  $f(x) = 0$  has the root  $r$ :  $f(r) = 0$ .

Denote  $\Delta r$ :  $\Delta r = r - \bar{r}$  or  $r = \bar{r} + \Delta r$ ,  $\Delta r$  is the forward error.

Denote  $\Delta f$ :  $\Delta f = f - \bar{f}$  or  $f = \bar{f} + \Delta f$ ,  $\Delta f$  is the backward error.

$$(f + \Delta f)(r + \Delta r) = 0 \Leftrightarrow f(r + \Delta r) + \Delta f(r + \Delta r) = 0$$

We apply Taylor series, ignoring second order terms:

$$\begin{aligned}f(r + \Delta r) &= f(r) + f'(r)\Delta r + \dots \\ \Delta f(r + \Delta r) &= \Delta f(r) + \Delta f'(r)\Delta r + \dots\end{aligned}$$

Note that  $f(r) = 0$  and  $\Delta f'(r)\Delta r$  is of second order.

$$(f + \Delta f)(r + \Delta r) = 0 \Leftrightarrow f'(r)\Delta r + \Delta f(r) \approx 0.$$

### Theorem (sensitivity of a root)

For an equation  $f(x) = 0$  with root  $r$ :  $|\Delta r| \approx \left| \frac{\Delta f(r)}{f'(r)} \right|$ .

## a first exercise

### Exercise 1:

Consider  $p(x) = (x - 1.501)(x - 1.5)(x - 1.499)$ .

- 1 Compute the forward and backward error on the root  $r = 1.5$  of  $p$ .
- 2 Evaluate the formula for the sensitivity of  $r$ .
- 3 If we want four correct decimal places in a root, how high should the precision be at which we evaluate the coefficients of  $p$ ?

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# The Wilkinson Polynomial

The roots of the Wilkinson polynomials are consecutive integers  $1, 2, \dots$ . For example, the 20-th Wilkinson polynomial is

$$p(x) = (x - 1)(x - 2) \cdots (x - 20).$$

To test a root finder, this polynomial seems like an ideal test problem.

When expanded, the constant coefficient of  $p$  is

$$20! = 2432902008176640000$$

and  $20! = 2.43290200817664e18$  as a 64-bit float.

Evaluating  $p$  correctly is the main difficulty.

## evaluating Wilkinson polynomials

```
julia> import Pkg; Pkg.add("SymPy")
julia> using SymPy
julia> x = Sym("x")
julia> w20 = prod([(x - Float64(r)) for r = 1:20])
julia> e20 = expand(w20)
```

Observe the conversion `Float64(r)` of the integer `r`.

```
julia> subs(w20, x => 1.0)
0
julia> subs(e20, x => 1.0)
-2560.000000000000
```

### Exercise 2:

Do the above calculation for the other 19 roots of the 20-th Wilkinson polynomial. What do you observe?

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# condition number

Every problem in numerical analysis has a condition number.

## Definition (condition number of a numerical problem)

For a numerical problem with input  $x$  and output  $y$ ,

*the condition number*  $\kappa$  is  $\frac{|\Delta y|}{|y|} \leq \kappa \frac{|\Delta x|}{|x|}$ .

The condition number measures how errors in the input are magnified to errors in the output.

A problem is

- *well-conditioned* if  $\kappa$  is small, and
- *ill-conditioned* if  $\kappa$  is large.

If  $\kappa = \infty$  (in case of multiple roots), then the problem is *ill-posed*.

## application to the root finding problem

For an equation  $f(x) = 0$  with root  $r$ ,  
we derived the sensitivity formula for the root:

$$|\Delta r| \approx \left| \frac{\Delta f(r)}{f'(r)} \right|.$$

To derive the condition number, divide both sides by  $|r|$  and insert  $|f|$ :

$$\left| \frac{\Delta r}{r} \right| \approx \frac{1}{|f'(r)|} \left| \frac{\Delta f(r)}{r} \right| = \frac{|f|}{|r f'(r)|} \left| \frac{\Delta f(r)}{f} \right|,$$

where  $|f|$  measures the size (of the coefficients) of  $f$ .

### Theorem (condition number of the root finding problem)

*The equation  $f(x) = 0$  with root  $r$  has condition number  $\kappa = \frac{|f|}{|r f'(r)|}$ .*

# interpretation of the condition number

For an equation  $f(x) = 0$  with root  $r$ , the condition number is

$$\kappa = \frac{|f|}{|r f'(r)|}.$$

Three factors in the magnification of the backward error:

- 1  $|f|$  is the size of  $f$ , some examples are
  - ▶ the largest coefficient of  $f$  if a polynomial, or
  - ▶ the largest value  $f(x)$  can take in the neighborhood of  $r$ .

This factor captures the numerical difficulty of evaluating  $f$ .

- 2  $1/|r|$ : the smaller the root, the larger the relative error.
- 3  $1/|f'(r)|$ : if  $r$  is close to a multiple root, then  $f'(r) \approx 0$ .

# the Wilkinson polynomials again

## Exercise 3:

Consider  $p(x) = \prod_{r=1}^d (x - r)$ , for some finite degree  $d$ .

Evaluate the formula  $\kappa = \frac{|p|}{|r p'(r)|}$  at the first root  $r = 1$  of  $p$ ,

for increasing values of the degree  $d = 2, 3, \dots, 20$ ,  
using the absolute value of the largest coefficient of  $p$   
as the value for  $|p|$ .

For which value of  $d$  does  $\kappa$  become larger than  $10^8$ ?