Runge-Kutta Methods

1. Local and Global Errors
   - truncation of Taylor series
   - errors of Euler’s method and the modified Euler method

2. Runge-Kutta Methods
   - derivation of the modified Euler method
   - application on the test equation
   - third and fourth order Runge-Kutta methods

3. Applications
   - the pendulum problem
   - the 3-body problem in celestial mechanics

MCS 471 Lecture 30
Numerical Analysis
Jan Verschelde, 1 November 2021
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local and global errors

We consider the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$ 

For some step $h > 0$, we set $x_1 = x_0 + h$ and compute

$$y(x_0 + h) = y(x_0) + y'(x_0)h + y''(x_0)\frac{h^2}{2!} + \cdots + O(h^p).$$

The order of the method is $p$ if the approximation coincides with the first $p$ terms of the Taylor series.

The *local error* of a method is the error of one step: $|y_1 - y(x_1)|$.

In the *global error* we take the accumulation of errors into account. After $n$ steps, $x_n = x_0 + nh$, and the global error is $|y_n - y(x_n)|$.

If the local error is $O(h^p)$, then the global error is $O(h^{p-1})$. 
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Euler’s method and the modified Euler method

Consider two examples:

1. Euler’s method: \[ y_{n+1} = y_n + hf(x_n, y_n), \quad n = 0, 1, \ldots \]
   Local error: \( O(h^2) \), global error: \( O(h) \).

2. the modified Euler method:

   \[
   \bar{y}_{n+1} = y_n + hf(x_n, y_n) \\
   \quad y_{n+1} = y_n + \frac{h}{2} \left( f(x_n, y_n) + f(x_{n+1}, \bar{y}_{n+1}) \right), \quad n = 0, 1, \ldots
   \]

   Local error: \( O(h^3) \), global error: \( O(h^2) \).
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Consider a 2-stage Runge-Kutta method:

\[
\begin{align*}
    k_1 &= f(x_n, y_n) \\
    k_2 &= f(x_n + \alpha h, y_n + \beta k_1) \\
    y_{n+1} &= y_n + ak_1 + bk_2.
\end{align*}
\]

This is a 2-stage method because we have 2 function evaluations.

There are four parameters: \(\alpha, \beta, a, b\).

The goal is to determine \(\alpha, \beta, a, b\) so that the order of the method is as high as possible.
a 2-stage Runge-Kutta method

\[
\begin{align*}
  k_1 &= f(x_n, y_n) \\
  k_2 &= f(x_n + \alpha h, y_n + \beta k_1) \\
  y_{n+1} &= y_n + a k_1 + b k_2
\end{align*}
\]

is equivalent to

\[
y_{n+1} = y_n + a f(x_n, y_n) + b f(x_n + \alpha h, y_n + \beta f(x_n, y_n)).
\]

We apply Taylor series in several variables:

\[
f(x_n + \alpha h, y_n + \beta f(x_n, y_n)) \\
\approx f(x_n, y_n) + \alpha h f_x(x_n, y_n) + \beta f(x_n, y_n) f_y(x_n, y_n)
\]

which becomes

\[
y_{n+1} = y_n + a f(x_n, y_n) \\
+ b \left( f(x_n, y_n) + \alpha h f_x(x_n, y_n) + \beta f(x_n, y_n) f_y(x_n, y_n) \right).
\]
compare the terms with Taylor series of $y(x_n + h)$

$$y_{n+1} = y_n + af(x_n, y_n) + b\left(f(x_n, y_n) + \alpha hf_x(x_n, y_n) + \beta f(x_n, y_n)f_y(x_n, y_n)\right)$$

$$y_{n+1} = y_n + (a + b)f(x_n, y_n) + \alpha bh f_x(x_n, y_n) + b\beta f(x_n, y_n)f_y(x_n, y_n)$$

Develop $y(x_n + h)$ at $x_n$ with Taylor series:

$$y(x_n + h) = y(x_n) + hy'(x_n) + \frac{h^2}{2}y''(x_n) + O(h^3).$$

We apply this to the differential equation $y' = f$:

$$y_{n+1} = y_n + hf(x_n, y_n) + \frac{h^2}{2} \frac{d}{dx} f(x, y(x)) \bigg|_{(x_n, y_n)}.$$
conditions on the parameters

\[ y_{n+1} = y_n + hf(x_n, y_n) + \frac{h^2}{2} \left. \frac{d}{dx} f(x, y(x)) \right|_{(x_n, y_n)} \]

\[ \frac{d}{dx} f(x, y(x)) = f_x(x, y(x)) + f_y(x, y(x))y'(x), \quad y'(x) = f(x, y(x)) \]

\[ \left. \frac{d}{dx} f(x, y(x)) \right|_{(x_n, y_n)} = f_x(x_n, y_n) + f_y(x_n, y_n)f(x_n, y_n) \]

\[ y_{n+1} = y_n + hf(x_n, y_n) + \frac{h^2}{2} \left( f_x(x_n, y_n) + f_y(x_n, y_n)f(x_n, y_n) \right) \]

Compare this to

\[ y_{n+1} = y_n + (a + b)f(x_n, y_n) + \alpha bh f_x(x_n, y_n) + b\beta f(x_n, y_n)f_y(x_n, y_n). \]
determination of the parameters

The parameters in the 2-stage Runge-Kutta method

\[ y_{n+1} = y_n + (a + b)f(x_n, y_n) + \alpha bh f_x(x_n, y_n) + b\beta f(x_n, y_n)f_y(x_n, y_n) \]

is compared with the Taylor series of the solution at \( x_n + h \):

\[ y_{n+1} = y_n + hf(x_n, y_n) + \frac{h^2}{2} \left( f_x(x_n, y_n) + f_y(x_n, y_n)f(x_n, y_n) \right) . \]

The system on \( \alpha, \beta, a, b \) is

\[
\begin{cases}
  a + b &= h \\
  \alpha bh &= h^2/2 \\
  b\beta &= h^2/2 .
\end{cases}
\]

We can solve this system symbolically with SymPy.
solving \( a + b = h, \ \alpha bh = h^2/2, \ b\beta = h^2/2 \)

using SymPy

\[ a, b, \alpha, \beta, h = \text{Sym}("a, b, \alpha, \beta, h") \]

\[ \text{eq1} = a + b - h \]
\[ \text{eq2} = \alpha \cdot b \cdot h - h^2/2 \]
\[ \text{eq3} = b \cdot \beta - h^2/2 \]

\[ \text{sys} = \{\text{eq1}, \text{eq2}, \text{eq3}\} \]

\[ \text{sol} = \text{solve}(\text{sys}, [a, b, \alpha, \beta]) \]

println(sol)

The output is a 4-tuple of symbolic values for \( a, b, \alpha, \text{and} \ \beta \):

\[ (\frac{-h \cdot (-2 \beta + h)}{2 \beta}, \frac{h^2}{2 \beta}, \frac{\beta}{h}, \beta) \]

Choose \( \beta = h \), then \( \alpha = 1 \), and \( a = \frac{h}{2} = b \).
the modified Euler method

The solution on the previous slide leads to

\[ y_{n+1} = y_n + \frac{h}{2} \left( f(x_n, y_n) + f(x_n + h, y_n + h f(x_n, y_n)) \right), \]

which allows to rewrite the modified Euler method as a 2-stage Runge-Kutta method:

\[
\begin{align*}
    k_1 &= f(x_n, y_n) \\
    k_2 &= f(x_n + h, y_n + h k_1) \\
    y_{n+1} &= y_n + \frac{h}{2} \left( k_1 + k_2 \right).
\end{align*}
\]

This shows that the modified Euler method has order 3, which is equivalent to stating that the local error is \( O(h^3) \).
The modified Euler method is one of the rules of the form

\[ y_{n+1} = y_n + (a + b)f(x_n, y_n) + \alpha bh f_x(x_n, y_n) + b\beta f(x_n, y_n)f_y(x_n, y_n) \]

where the parameters satisfy

\[
\begin{align*}
    a + b &= h \\
    \alpha bh &= h^2/2 \\
    b\beta &= h^2/2.
\end{align*}
\]

Exercise 1:
Set \( \alpha = 1/2 \) and derive the midpoint method. Verify the order of the midpoint method.
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application on the test equation

\[ y(x) = \exp(x) \] is the exact solution of the test equation:

\[ \frac{dy}{dx} = y, \quad y(0) = 1. \]

Apply the 2-stage Runge-Kutta method

\[
\begin{align*}
    k_1 &= f(x_n, y_n) \\
    k_2 &= f(x_n + h, y_n + h k_1) \\
    y_{n+1} &= y_n + \frac{h}{2} \left( k_1 + k_2 \right)
\end{align*}
\]

with \( f = y \):

\[
\begin{align*}
    k_1 &= y_n \\
    k_2 &= y_n + h k_1 \\
    y_{n+1} &= y_n + \frac{h}{2} \left( k_1 + k_2 \right)
\end{align*}
\]
a Julia function

""
      rk2exp(n::Int64)
""

A 2-stage Runge-Kutta method with n steps
on the interval [0,1] on y' = y, y(0) = 1.
""

function rk2exp(n::Int64)
    h = 1.0/n
    y0 = 1.0
    y1 = 1.0
    for i=1:n
        x = i*h
        k1 = y0
        k2 = y0 + h*k1
        y1 = y0 + (h/2)*(k1 + k2)
        y0 = y1
    end
    return y1
end
on the test equation with \( h = 1/10 \)

Running a 2-stage Runge-Kutta method ...

<table>
<thead>
<tr>
<th>i</th>
<th>x</th>
<th>k2</th>
<th>2-stage RK</th>
<th>exact</th>
<th>error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.10</td>
<td>1.100000e+00</td>
<td>1.105000e+00</td>
<td>1.105171e+00</td>
<td>1.71e-04</td>
</tr>
<tr>
<td>2</td>
<td>0.20</td>
<td>1.215500e+00</td>
<td>1.221025e+00</td>
<td>1.221403e+00</td>
<td>3.78e-04</td>
</tr>
<tr>
<td>3</td>
<td>0.30</td>
<td>1.343128e+00</td>
<td>1.349233e+00</td>
<td>1.349859e+00</td>
<td>6.26e-04</td>
</tr>
<tr>
<td>4</td>
<td>0.40</td>
<td>1.484156e+00</td>
<td>1.490902e+00</td>
<td>1.491825e+00</td>
<td>9.23e-04</td>
</tr>
<tr>
<td>5</td>
<td>0.50</td>
<td>1.639992e+00</td>
<td>1.647447e+00</td>
<td>1.648721e+00</td>
<td>1.27e-03</td>
</tr>
<tr>
<td>6</td>
<td>0.60</td>
<td>1.812191e+00</td>
<td>1.820429e+00</td>
<td>1.822119e+00</td>
<td>1.69e-03</td>
</tr>
<tr>
<td>7</td>
<td>0.70</td>
<td>2.002472e+00</td>
<td>2.011574e+00</td>
<td>2.013753e+00</td>
<td>2.18e-03</td>
</tr>
<tr>
<td>8</td>
<td>0.80</td>
<td>2.212731e+00</td>
<td>2.222789e+00</td>
<td>2.225541e+00</td>
<td>2.75e-03</td>
</tr>
<tr>
<td>9</td>
<td>0.90</td>
<td>2.445068e+00</td>
<td>2.456182e+00</td>
<td>2.459603e+00</td>
<td>3.42e-03</td>
</tr>
<tr>
<td>10</td>
<td>1.00</td>
<td>2.701800e+00</td>
<td>2.714081e+00</td>
<td>2.718282e+00</td>
<td>4.20e-03</td>
</tr>
</tbody>
</table>

This the same output as the modified Euler method:

- the local error is \(1.71e-04\),
- the global error is \(4.20e-03\).
application of the midpoint method

Exercise 2:
Apply the midpoint method (see Exercise 1) to the test equation. Compare the local and global error of the midpoint method to the output of the modified Euler method.
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A $p$-th order Runge-Kutta method proceeds in $p$ stages:

\[
\begin{align*}
    k_1 &= f(x_n, y_n) \\
    k_2 &= f(x_n + \alpha_2 h, y_n + \beta_2 k_1) \\
    k_3 &= f(x_n + \alpha_3 h, y_n + \beta_3 k_2) \\
    &\vdots \\
    k_p &= f(x_n + \alpha_p h, y_n + \beta_p k_{p-1}) \\
    y_{n+1} &= y_n + a_1 k_1 + a_2 k_2 + a_3 k_3 + \cdots + a_p k_p.
\end{align*}
\]

If the parameters $\alpha_2, \alpha_3, \ldots, \alpha_p, \beta_2, \beta_3, \ldots, \beta_p$, and $a_1, a_2, a_3, \ldots, a_p$ are determined to agree with the Taylor series of $y(x_n + h)$, then the local error is $O(h^{p+1})$, the global error is $O(h^p)$. 

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A third order Runge-Kutta method proceeds in three stages:

\[
\begin{align*}
  k_1 &= f(x_n, y_n) \\
  k_2 &= f(x_n + h/2, y_n + hk_1/2) \\
  k_3 &= f(x_n + 3h/4, y_n + 3hk_2/4) \\
  y_{n+1} &= y_n + \frac{h}{9} \left( 2k_1 + 3k_2 + 4k_3 \right).
\end{align*}
\]

The local error is $O(h^4)$, the global error is $O(h^3)$. 

on the test equation with \( h = 1/10 \)

Running a 3-stage Runge-Kutta method ...

<table>
<thead>
<tr>
<th>i</th>
<th>x</th>
<th>k3</th>
<th>3-stage RK</th>
<th>exact</th>
<th>error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>1.105167e+00</td>
<td>1.105171e+00</td>
<td>4.25e-06</td>
</tr>
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<td>1.221393e+00</td>
<td>1.221403e+00</td>
<td>9.40e-06</td>
</tr>
<tr>
<td>3</td>
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<td>1.317578e-01</td>
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<td>1.56e-05</td>
</tr>
<tr>
<td>4</td>
<td>0.40</td>
<td>1.456143e-01</td>
<td>1.491802e+00</td>
<td>1.491825e+00</td>
<td>2.30e-05</td>
</tr>
<tr>
<td>5</td>
<td>0.50</td>
<td>1.609281e-01</td>
<td>1.648690e+00</td>
<td>1.648721e+00</td>
<td>3.17e-05</td>
</tr>
<tr>
<td>6</td>
<td>0.60</td>
<td>1.778524e-01</td>
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<td>1.822119e+00</td>
<td>4.21e-05</td>
</tr>
<tr>
<td>7</td>
<td>0.70</td>
<td>1.965565e-01</td>
<td>2.013698e+00</td>
<td>2.013753e+00</td>
<td>5.42e-05</td>
</tr>
<tr>
<td>8</td>
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<td>2.225472e+00</td>
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<td>1.00</td>
<td>2.653205e-01</td>
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<td>2.718282e+00</td>
<td>1.05e-04</td>
</tr>
</tbody>
</table>

For this 3-stage Runge-Kutta method,
- the local error is \( 4.25 \times 10^{-6} \),
- the global error is \( 1.05 \times 10^{-4} \).
A fourth order Runge-Kutta method proceeds in four stages:

\[
\begin{align*}
k_1 &= f(x_n, y_n) \\
k_2 &= f(x_n + h/2, y_n + (h/2)k_1) \\
k_3 &= f(x_n + h/2, y_n + (h/2)k_2) \\
k_4 &= f(x_n + h, y_n + hk_3) \\
y_{n+1} &= y_n + \frac{h}{6} \left( k_1 + 2k_2 + 2k_3 + k_4 \right).
\end{align*}
\]

The local error is \(O(h^5)\), the global error is \(O(h^4)\).
on the test equation with $h = 1/10$

Running a 4-stage Runge-Kutta method ...

<table>
<thead>
<tr>
<th>i</th>
<th>x</th>
<th>k4</th>
<th>4-stage RK</th>
<th>exact</th>
<th>error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.10</td>
<td>1.105250e-01</td>
<td>1.105171e+00</td>
<td>1.105171e+00</td>
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</tr>
<tr>
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<td>1.221490e-01</td>
<td>1.221403e+00</td>
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<tr>
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<td>0.40</td>
<td>1.491931e-01</td>
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<td>4.58e-07</td>
</tr>
<tr>
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<td>1.648839e-01</td>
<td>1.648721e+00</td>
<td>1.648721e+00</td>
<td>6.32e-07</td>
</tr>
<tr>
<td>6</td>
<td>0.60</td>
<td>1.822248e-01</td>
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<td>2.013896e-01</td>
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</tr>
<tr>
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<td>2.225699e-01</td>
<td>2.225540e+00</td>
<td>2.225541e+00</td>
<td>1.37e-06</td>
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<tr>
<td>9</td>
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<td>2.459778e-01</td>
<td>2.459601e+00</td>
<td>2.459603e+00</td>
<td>1.70e-06</td>
</tr>
<tr>
<td>10</td>
<td>1.00</td>
<td>2.718474e-01</td>
<td>2.718280e+00</td>
<td>2.718282e+00</td>
<td>2.08e-06</td>
</tr>
</tbody>
</table>

For this 4-stage Runge-Kutta method,

- the local error is $8.47e-08$,
- the global error is $2.08e-06$. 
summary of the experiments

Running Runge-Kutta methods of order 2, 3, and 4 on the test equation \( y' = y, \ y(0) = 1 \).

On the interval \([0, 1]\), we do \( n = 10 \) steps, \( h = 1/n = 1/10 \).

For a \( p \)-stage Runge-Kutta method, we expect

- a local error of \( O(h^{p+1}) \), and
- a global error of \( O(h^p) \).

The actual values for local and global errors are below:

<table>
<thead>
<tr>
<th>( p )</th>
<th>local error</th>
<th>global error</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.71e-04</td>
<td>4.20e-03</td>
</tr>
<tr>
<td>3</td>
<td>4.25e-06</td>
<td>1.05e-04</td>
</tr>
<tr>
<td>4</td>
<td>8.47e-08</td>
<td>2.08e-06</td>
</tr>
</tbody>
</table>

The values agree with the expectations.
the test equation with a parameter

For some parameter $\lambda$, the initial value problem

$$\frac{dy}{dx} = \lambda y, \quad y(0) = 1$$

has $y(x) = \exp(\lambda x)$ as the exact solution.

Exercise 3:

1. Take $\lambda = 0.1$ and consider the interval $[0, 1]$. For $h = 0.1$, run the 2-stage Runge-Kutta method. Compare the local and global error with the errors on the test equation without parameter (or for $\lambda = 1$).

2. Take $\lambda = 0.01$ and consider the interval $[0, 1]$. For $h = 0.1$, run the 4-stage Runge-Kutta method. Compare the results with your findings of the previous part.
Runge-Kutta Methods

1. Local and Global Errors
   - truncation of Taylor series
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2. Runge-Kutta Methods
   - derivation of the modified Euler method
   - application on the test equation
   - third and fourth order Runge-Kutta methods

3. Applications
   - the pendulum problem
   - the 3-body problem in celestial mechanics
the pendulum problem

The initial value problem to model a pendulum is

\[ y'_1 = y_2, \]
\[ y'_2 = -\frac{g}{\ell} \sin(y_1), \quad y_1 = \pi/4, \quad y_2 = 0. \]

We apply a vector version of a 2-stage Runge-Kutta method:

\[
\begin{cases}
  k_1 = f(x_n, y_n) \\
  k_2 = f(x_n + h, y_n + h k_1) \\
  y_{n+1} = y_n + \frac{h}{2} (k_1 + k_2)
\end{cases}
\]

where

\[
y_n = \begin{bmatrix} y_{n,1} \\ y_{n,2} \end{bmatrix}, \quad k_1 = \begin{bmatrix} k_{1,1} \\ k_{1,2} \end{bmatrix}, \quad k_2 = \begin{bmatrix} k_{2,1} \\ k_{2,2} \end{bmatrix}, \quad f = \begin{bmatrix} y_2 \\ (-g/\ell) \sin(y_1) \end{bmatrix}.
\]
a 2-stage Runge-Kutta method for the pendulum

\[
y_n = \begin{bmatrix} y_{n,1} \\ y_{n,2} \end{bmatrix}, \quad k_1 = \begin{bmatrix} k_{1,1} \\ k_{1,2} \end{bmatrix}, \quad k_2 = \begin{bmatrix} k_{2,1} \\ k_{2,2} \end{bmatrix}, \quad f = \begin{bmatrix} y_2 \\ (-g/\ell)\sin(y_1) \end{bmatrix}
\]

\[
\begin{align*}
k_1 &= f(x_n, y_n) \\
k_2 &= f(x_n + h, y_n + h k_1) \\
y_{n+1} &= y_n + \frac{h}{2} \left( k_1 + k_2 \right)
\end{align*}
\]

\[
\begin{bmatrix} k_{1,1} \\ k_{1,2} \end{bmatrix} = \begin{bmatrix} y_2 \\ (-g/\ell)\sin(y_1) \end{bmatrix}, \quad \begin{bmatrix} k_{2,1} \\ k_{2,2} \end{bmatrix} = \begin{bmatrix} y_2 + h k_{1,2} \\ (-g/\ell)\sin(y_1 + h k_{1,1}) \end{bmatrix},
\]

\[
\begin{bmatrix} y_{n+1,1} \\ y_{n+1,2} \end{bmatrix} = \begin{bmatrix} y_{n,1} \\ y_{n,2} \end{bmatrix} + \frac{h}{2} \left( \begin{bmatrix} k_{1,1} \\ k_{1,2} \end{bmatrix} + \begin{bmatrix} k_{2,1} \\ k_{2,2} \end{bmatrix} \right)
\]
running 24 steps

$ julia rkpend.jl
Running the modified Euler method ...

\[
\begin{array}{cccc}
1 & 0.00 & 7.853982e-01 & 0.000000e+00 \\
25 & 6.28 & 1.550520e+00 & 2.841312e+00 \\
\end{array}
\]

Running a 2-stage Runge-Kutta method ...

\[
\begin{array}{cccc}
1 & 0.00 & 7.853982e-01 & 0.000000e+00 \\
25 & 6.28 & 1.550520e+00 & 2.841312e+00 \\
\end{array}
\]

Running a 3-stage Runge-Kutta method ...

\[
\begin{array}{cccc}
1 & 0.00 & 7.853982e-01 & 0.000000e+00 \\
25 & 6.28 & 5.106470e-01 & -7.919150e-01 \\
\end{array}
\]

Running a 4-stage Runge-Kutta method ...

\[
\begin{array}{cccc}
1 & 0.00 & 7.853982e-01 & 0.000000e+00 \\
25 & 6.28 & 7.543960e-01 & -1.190943e-01 \\
\end{array}
\]

The columns are respectively the step number \( n \), the value for \( t_n, y_{n,1} \) (position), \( y_{n,2} \) (velocity), and the error.

As expected, the errors decrease as the order increases.
24 steps with a 4-stage Runge-Kutta method
Exercise 4:
Apply Runge-Kutta methods of order three and four to

\[
\begin{align*}
\frac{dx}{dt} &= -y + \cos(t)\sin(t) \\
\frac{dy}{dt} &= x + \sin^2(t)
\end{align*}
\]

for \( t \in [0, 2\pi] \). Its exact solution is the cardioid.

1. For both the order three and four Runge-Kutta methods, set \( h \) to be small enough so the plot of the computed points agrees with the plot of the exact solution.

2. Compare the accuracy of both runs. What is the largest \( h \) you can use with the fourth order method and achieve the same accuracy as with the third order?
Exercise 5:
Apply Runge-Kutta methods of order three and four to
\[
\begin{align*}
\frac{dx}{dt} &= -y + 3 \cos(3t) \cos(t) \\
\frac{dy}{dt} &= x + 3 \cos(3t) \sin(t)
\end{align*}
\]
for \( t \in [0, 2\pi] \). See the previous lecture for its exact solution.

1. For both the order three and four Runge-Kutta methods, set \( h \) to be small enough so the plot of the computed points agrees with the plot of the exact solution.

2. Compare the accuracy of both runs. What is the largest \( h \) you can use with the fourth order method and achieve the same accuracy as with the third order?
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the 3-body problem

We consider three bodies with respective masses $m_1$, $m_2$, $m_3$ in the plane with positions $(x_1(t), y_1(t))$, $(x_2(t), y_2(t))$, $(x_3(t), y_3(t))$ evolving over time $t$, governed by a system of second order differential equations, shown below for the movement of the first body:

\[
\frac{d^2 x_1(t)}{dt^2} = -\frac{m_2(x_1(t) - x_2(t))}{((x_1(t) - x_2(t))^2 + (y_1(t) - y_2(t))^2)^{3/2}} - \frac{m_3(x_1(t) - x_3(t))}{((x_1(t) - x_3(t))^2 + (y_1(t) - y_3(t))^2)^{3/2}}
\]

The equation for $y_1(t)$ is similar (replace $x$ in numerator by $y$).

With four additional equations for the positions of the second and third body, our model consists of six second order equations.
a system of 12 first order equations

To turn this into a system of first order differential equations we introduce new variables $u_i, v_i$ for the velocities of $x_i, y_i$ so we have

\[
\frac{dx_1(t)}{dt} = u_1(t)
\]

\[
\frac{du_1(t)}{dt^2} = - \frac{m_2(x_1(t) - x_2(t))}{((x_1(t) - x_2(t))^2 + (y_1(t) - y_2(t))^2)^{3/2}} - \frac{m_3(x_1(t) - x_3(t))}{((x_1(t) - x_3(t))^2 + (y_1(t) - y_3(t))^2)^{3/2}}
\]

So we obtain 12 first order differential equations.

See the posted Julia program and Jupyter notebook.
a figure eight