MCS 471: Review of Chapters 3, 4, and 5

The questions below are meant to stimulate the preparation of the final exam, concerning chapters 3, 4, and 5. They cover some of the most important but not all topics. You must review all course materials: textbook, homework exercises, lecture notes (also consult the course web site), quizzes, and midterm exams. Also please review the computer projects.

1. Consider the polynomial $p(x) = x^2 + x - 5$.
   (a) Construct the Newton form of $p(x)$ by divided differences, using the points $(x_i, p(x_i))$, with $x_i = i$, for $i = 0, 1, 2, 3$.
   (b) Explain why the last element f_{0123} in the table of divided differences you constructed above is (or should have been) zero.
   (c) Apply Neville's algorithm to evaluate the interpolating polynomial at 0.5.
   (d) Approximate $p(x)$ with the linear function that minimizes the squares of the errors, using the points $(x_i, p(x_i))$, with $x_i = i$, for $i = 0, 1, 2, 3$.

2. The Maclaurin expansion of e^{x^2} is

$$e^{x^2} = 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \frac{1}{24}x^8 + O(x^{10})$$

   Use this Maclaurin expansion to construct a Padé approximation for e^{x^2} as a quotient of two quadrics.

3. The answer to the previous question is $r(x) = \frac{1+\frac{1}{2}x^2}{1-\frac{1}{2}x^2}$
   (a) Compute a continued-fraction representation of $r(x)$.
   (b) Count the number of arithmetical operations you need to evaluate the continued-fraction representation of $r(x)$. Compare this number with the number of arithmetical operations needed to evaluate $r(x)$ if you would use the Horner form of the numerator and denominator of $r(x)$.

4. Consider a function whose values are tabulated below:

x	$f(x)$
0.000	1.0000000000
0.125	0.9921976672
0.250	0.9689124217
0.375	0.9305076219
0.500	0.8775825619
0.625	0.8109631195
0.750	0.7316888689
0.875	0.6409968582
1.000	0.5403023059

   Suppose we are interested in the derivative of $f(x)$ at $x = 0.5$.
   (a) One way to compute the derivative would be to compute the interpolating polynomial through all these points and then to take its derivative for $x = 0.5$. Explain why numerically this is not such a good idea.
   (b) Compute the most accurate approximation for $f'(0.5)$. Estimate the accuracy in your answer.

5. Consider the operator D defined by $Df(x) = \frac{\partial f}{\partial x}$, and the operator Δ defined by $\Delta f(x) = f(x+h) - f(x)$, for any $h > 0$. Show that $D = \frac{1}{h}\ln(I + \Delta)$.

6. Suppose we interpolate $f(x)$ with divided differences (i.e., with Newton interpolation) in equidistant points, at points $x_i = x_0 + ih$, $i = 0, 1, \ldots, n$. Write $f[x_0, x_1, x_2]$ using the operator Δ.

7. Explain why the formulas for Richardson extrapolation to approximate the derivative with forward differences are different from those that use central differences. In particular, why do we see only even powers of the ratio r in the extrapolation formulas that use central differences?

8. Consider $\displaystyle\int_0^1 e^x \cos(2\pi x) dx$

   (a) Apply the composite trapezoidal rule using four subintervals of equal length to approximate this integral. Write your answer with six decimal places.

   (b) Estimate the accuracy of the numerical approximation you just computed. How many decimal places can be correct?

   (c) Apply Romberg integration to obtain a sixth-order approximation of the integral. Use six decimal places to denote the answers. What is the accuracy of your final answer?

9. Simpson's rule on an interval $[a, b]$ approximates $\displaystyle\int_a^b f(x)dx$ by $\frac{b-a}{6}\left(f(a) + 4f(\frac{a+b}{2}) + f(b)\right)$.

   (a) Write a composite formula to integrate $\displaystyle\int_a^b f(x)dx$ with Simpon's rule, using seven function evaluations.

   (b) Give the formula for the general composite Simpson's rule, over n subintervals of $[a, b]$, of length $h = \frac{b-a}{n}$.

10. Consider the approximation of $\displaystyle\int_0^{2a} f(x)dx$ by the rule $w_1 f(0) + w_2 f(\frac{a}{2}) + w_3 f(a)$.

   (a) Determine the weights w_1, w_2, and w_3 so that the rule has the highest possible algebraic degree of precision.

   (b) What is the highest possible algebraic degree of precision we can reach with three function evaluations? Give the nonlinear system in the weights w_1, w_2, w_3 and abscisses x_1, x_2, x_3 to determine the quadrature rule $w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3)$.

11. The Fourier series of a function $f(t)$ appear in two different forms: either as $f(t) = \dfrac{a_0}{2} + \displaystyle\sum_{k=1}^{\infty} a_k \cos(2\pi kt) + b_k \sin(2\pi kt)$ or as $f(t) = \displaystyle\sum_{k=-\infty}^{\infty} c_k e^{i2\pi kt}$.

    Derive the relationship between the coefficients c_k and (a_k, b_k).

FINAL EXAM is in BH 208 on Wednesday 4 May 2005, from 8 till 10AM.