Linear Programming

1. Optimization Problems
   - an objective subject to constraints
   - an application: the diet problem
2. The Simplex Algorithm
   - steps in the algorithm
   - a numerical example
   - operations research in Julia
   - duality
3. Proposals of Project Topics
   - some industrial applications
   - managing a production facility
   - network flow problems – polynomial time

MCS 472 Lecture 15
Industrial Math & Computation
Jan Verschelde, 16 February 2026

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Optimization Problems

maximize profit		minimize cost
subject to		subject to
$\mathrm{E}(\mathbf{x}) = \mathbf{a}$	*equalities*	$\mathrm{E}(\mathbf{x}) = \mathbf{a}$
$\mathrm{I}(\mathbf{x}) \leq \mathbf{b}$	*inequalities*	$\mathrm{I}(\mathbf{x}) \geq \mathbf{b}$

Definition

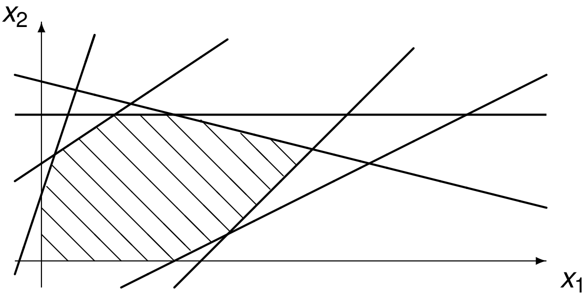
A *linear programming problem* is an optimization problem

- with one linear objective function; and
- all constraints (equalities and inequalities) are linear.

LP problem is short for linear programming problem.

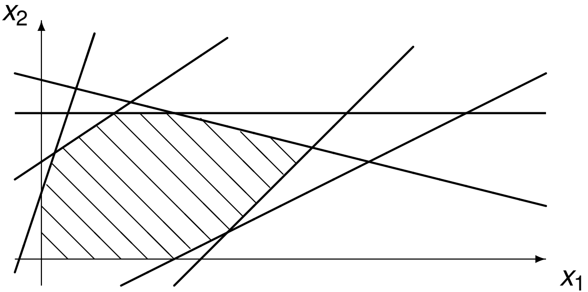
A Geometric View on Linear Programming

The linear constraints define a polyhedron:



- Polyhedron is not empty: the problem is *feasible*.
- Polyhedron is bounded: the problem is *well-posed*.

Convex Combinations



Definition (convex combination)

For points **x** and **y**, *a convex combination of* **x** *and* **y** is

$$\lambda \mathbf{x} + (1 - \lambda)\mathbf{y}, \text{ for } \lambda \in [0, 1].$$

A polyhedron P is *convex*: for any two points $\mathbf{x} \in P$ and $\mathbf{y} \in P$, the line segment connecting **x** and **y** lies entirely in P.

If **x** and **y** are solutions to an LP problem, then any convex combination of **x** and **y** is also a solution.

Standard Form for a Linear Minimization

$$\text{consider} \quad \begin{array}{ll} \min \mathbf{c}^T\mathbf{x} & \mathbf{c}, \mathbf{x} \in \mathbb{R}^n \\ \text{subject to } A\mathbf{x} \geq \mathbf{b} & A \in \mathbb{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^m \end{array}$$

Add slack variables $\mathbf{z} \in \mathbb{R}^m$:

$$A\mathbf{x} \geq \mathbf{b} \quad \text{becomes} \quad A\mathbf{x} - \mathbf{z} = \mathbf{b}, \ \mathbf{z} \geq \mathbf{0}$$

Split $\mathbf{x} = \mathbf{x}^+ - \mathbf{x}^-$, $\mathbf{x}^+ \geq \mathbf{0}$, $\mathbf{x}^- \geq \mathbf{0}$:

$$\min \ [\mathbf{c} \ \ -\mathbf{c} \ \ \mathbf{0}] \begin{bmatrix} \mathbf{x}^+ \\ \mathbf{x}^- \\ \mathbf{z} \end{bmatrix}$$

$$\text{subject to } [A \ \ -A \ \ -I] \begin{bmatrix} \mathbf{x}^+ \\ \mathbf{x}^- \\ \mathbf{z} \end{bmatrix} = \mathbf{b}, \qquad \begin{bmatrix} \mathbf{x}^+ \\ \mathbf{x}^- \\ \mathbf{z} \end{bmatrix} \geq \mathbf{0}$$

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Application: the Diet Problem

Problem: plan healthy but economical meals.

nutritional elements	food quantities x_1	x_2	$\cdots$	x_n	minimum requirements
1	a_{11}	a_{12}	$\cdots$	a_{1n}	c_1
2	a_{21}	a_{22}	$\cdots$	a_{2n}	c_2
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
m	a_{m1}	a_{m2}	$\cdots$	a_{mn}	c_m
price	p_1	p_2	$\cdots$	p_n	

Solve a linear programming problem (typically $m > n$) :

$$\min \ p_1 x_1 + p_2 x_2 + \cdots + p_n x_n$$

subject to

$$a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n \geq c_i, \ i = 1, 2, \ldots, m$$

a first exercise

Exercise 1:
Suppose General Motors makes a profit of \$100 on each Chevrolet, \$200 on each Buick, and \$400 on each Cadillac.

These cars get 20, 17, and 14 miles a gallon respectively.

It takes respectively 1, 2, and 3 minutes to assemble one Chevrolet, one Buick, and one Cadillac.

Assume the company is mandated by the government that the average car has a fuel efficiency of at least 18 miles a gallon.

Under these constraints, determine the optimal number of cars, maximizing the profit, which can be assembled in one 8-hour day.

Formulate the linear programming problem.

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The Simplex Algorithm

Idea: move from one vertex to another vertex, improving the objective value in each move.

Step 0: Bring the problem in standard form:
$\min \mathbf{c}^T\mathbf{x}$ subject to $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$.

Step 1: Find one feasible point.

Step 2: Find one feasible *vertex* point.

Step 3: Move to vertex with improved objective value, repeat until at optimum.

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a numerical example

Consider

$$\max 3x_1 + 5x_2$$

subject to

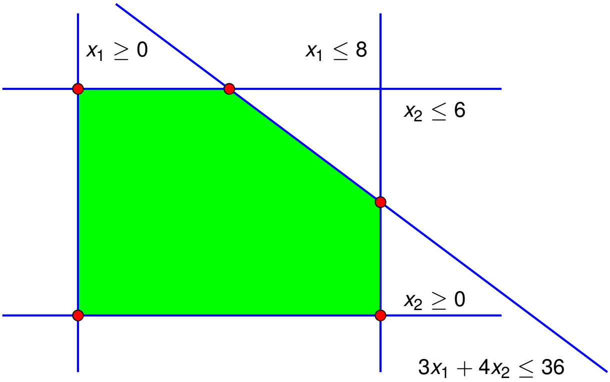
$$\left\{ \begin{array}{rcl} x_1 & \leq & 8 \\ x_2 & \leq & 6 \\ 3x_1 + 4x_2 & \leq & 36 \\ x_1 & \geq & 0 \\ x_2 & \geq & 0 \end{array} \right.$$

Imagine a production plan:

- x_1 can be sold at \$3, while x_2 yields \$5 per unit.
- We have constraints on labor resources, storage capacity, etc.

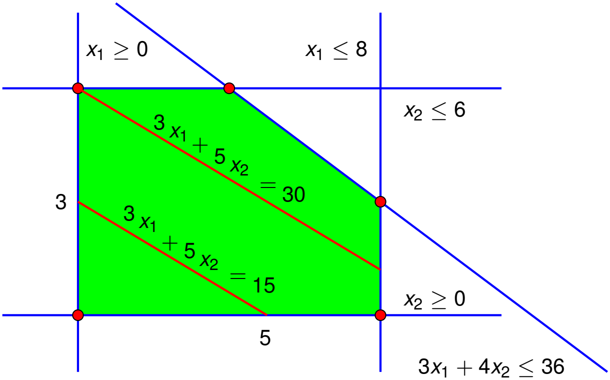
the feasible region

The constraints $x_1 \geq 0$, $x_2 \geq 0$, $x_1 \leq 8$, $x_2 \leq 6$, $3x_1 + 4x_2 \leq 36$ are represented by the blue lines:



the objective

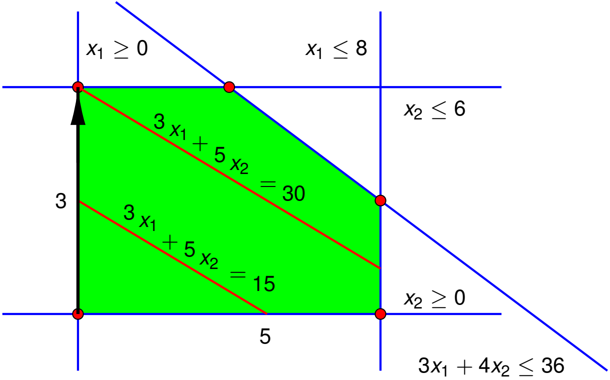
The objective $3x_1 + 5x_2$ is represented by the red line:



All points on the red line have the same value of the objective.

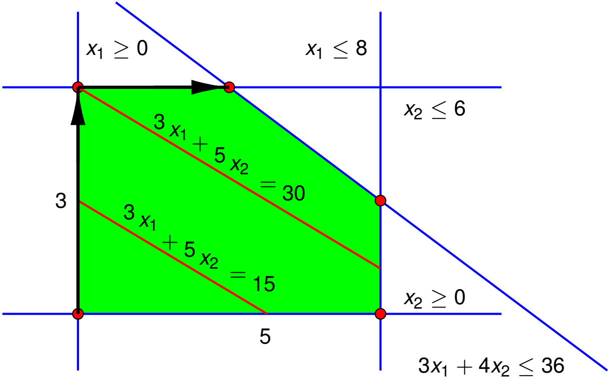
moving towards the optimal solution

Because the coefficient of x_2 in the objective $3x_1 + 5x_2$
is higher than the coefficient of x_1, we move from (0,0) to (0,6).



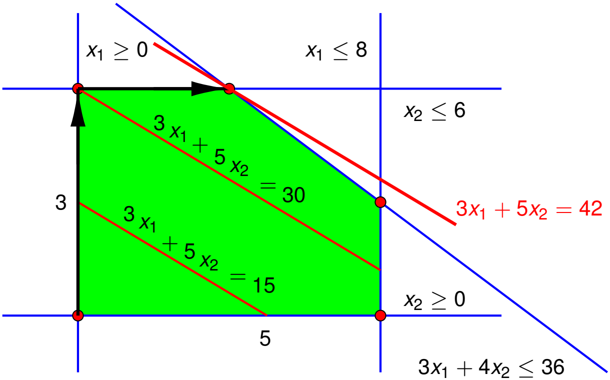
moving towards the optimal solution

Because of $x_2 \leq 6$, we cannot increase x_2,
but we can raise x_1 to 4, so we move from (0,6) to (4,6).



at the optimal solution

At (4,6), we cannot increase the value of the objective $3x_1 + 5x_2$.



At the optimum, the line $3x_1 + 5x_2 = 42$ passes through (4,6), the highest point in the feasible region.

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Operations Research in Julia

- JuMP is a modeling language and collection of packages for mathematical optimization in Julia.
- GLPK is used for linear programming. GLPK is the GNU Linear Programming Kit, free software.
- The first tutorial is an example of the diet problem. See the Jupyter notebook posted with this lecture.
- Changhyun Kwon: Julia Programming for Operations Research, second edition. Free to read online, examples on github.

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Every LP problem has a dual problem

$$\max \sum_{j=1}^{n} c_j x_j \quad \text{subject to} \quad \left\{ \begin{array}{l} \displaystyle\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \\ \qquad\qquad i=1,2,\ldots,m \\ x_j \geq 0, \quad j=1,2,\ldots,n \end{array} \right.$$

$$\min \sum_{i=1}^{m} b_i y_i \quad \text{subject to} \quad \left\{ \begin{array}{l} \displaystyle\sum_{j=1}^{m} a_{ji} y_j \geq c_i, \\ \qquad\qquad i=1,2,\ldots,n \\ y_j \geq 0, \quad j=1,2,\ldots,m \end{array} \right.$$

economic interpretation :

x_j	:	level of activity j
c_j	:	unit profit of activity j
b_i	:	amount available of resource i
a_{ij}	:	amount of resource i consumed by each unit of activity j
y_i	:	contribution to profit per unit of resource i

the dual of our numerical example

The dual of

$$\max 3x_1 + 5x_2 \\ \text{subject to} \\ \left\{ \begin{array}{rcl} x_1 & \leq & 8 \\ x_2 & \leq & 6 \\ 3x_1 + 4x_2 & \leq & 36 \\ x_1 & \geq & 0 \\ x_2 & \geq & 0 \end{array} \right.$$

is

$$\min 8y_1 + 6y_2 + 36y_3 \\ \text{subject to} \\ \left\{ \begin{array}{rcl} y_1 + 3y_3 & \geq & 3 \\ y_2 + 4y_3 & \geq & 5 \\ y_1 & \geq & 0 \\ y_2 & \geq & 0 \\ y_3 & \geq & 0 \end{array} \right.$$

the dual of the diet problem

Exercise 2:
See the posted notebook for an example of the diet problem, or, equivalently, look at the tutorial example of JuMP.

1. Write the dual of that example of the diet problem.
2. What are the units of the dual variables?

a third exercise

Exercise 3:
Suppose an investor has a choice between three types of shares.

Type A pays 4%, type B pays 6%, and type C pays 9% interest.

The investor has $100,000 available to buy shares and wants to maximize the interest, under the following constraints:

1. no more than $20,000 can be spent on shares of type C, and
2. at least $10,000 should be spent on shares of type A.

Answer the following questions:

1. Give the mathematical description of the optimization problem.
2. Bring the problem into the standard form.
3. Solve the linear programming problem.

weighted matching

Suppose we have three machines m_1, m_2, m_3, and three jobs j_1, j_2, j_3. Each machine is capable to do all jobs, but the completion times are different for each machine, as expressed in the table below:

	jobs		
machine	j_1	j_2	j_3
m_1	4	3	9
m_2	3	2	6
m_3	9	5	7

The first row states that machine m_1 does job j_1 in 4 minutes, job j_2 in 3 minutes, and job j_3 in 9 minutes. Read the other rows similarly. The problem is to assign each machine to exactly one job, minimizing the sum of the completion times.

Exercise 4: Formulate this problem as a linear programming problem. Solve the problem. Verify that your formulation is correct.

How good is the simplex algorithm?

The simplex algorithms performs very well on large problems, but in some cases an exponential number of vertices are visited, exponential in the number of equations and variables.

Exercise 5: Consider the sequences of vertices:

$$(\epsilon, \epsilon^2, \epsilon^3), (1, \epsilon, \epsilon^2), (1, 1-\epsilon, \epsilon-\epsilon^2), (\epsilon, 1-\epsilon^2, \epsilon-\epsilon^3),$$
$$(\epsilon, 1-\epsilon^2, 1-\epsilon+\epsilon^3), (1, 1-\epsilon, 1-\epsilon+\epsilon^2), (1, \epsilon, 1-\epsilon^2),$$
$$(\epsilon, \epsilon^2, 1-\epsilon^3), \text{ for } \epsilon : 0 < \epsilon < 1/2.$$

1. Verify that maximizing the third coordinate x_3, the vertices form an increasing sequence and that we visit all eight.
2. Set up the linear programming problem and solve it.

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1. some industrial applications

Read the paper by Ernest Koenigsberg: *Some Industrial Applications of Linear Programming*. Operational Research Society, Vol. 12, No. 2, pages 105–114, 1961.

1. Select one problem and explain the LP formulation, *in your own words.*
2. Work out an example with numerical data.
3. Use JuMP to solve an instance of the same dimensions as in the paper.
4. Report on the computational cost of the problem.

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2. managing a production facility

Read the first section of the book by Robert J. Vanderbei:
Linear Programming. Foundations and Extensions. Fifth Edition, Springer-Verlag, 2020.

1. Explain why the management of a production facility translates into an LP problem, *in your own words.*
2. Setup a program to generate numerical data, for any dimension.
3. Use JuMP to solve the generated instances.
4. Report on the computational cost for various dimensions.

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3. solving network flow problems with JuMP

A network is a weighted directed graph allowing for flow to go from a source to a sink along the edges which each have a capacity.

1. Starting with the JuMP tutorial on network flow problems, explore shortest path, assignment, and max-flow problems.
2. Read the paper by
   Shuvomoy Das Gupta, J. Kevin Tobin, and Lacra Pavel on
   A two-step linear programming model for energy-efficient timetables in metro railway networks,
   in *Transportation Research Part B*, vol. 93, pages 57-74, 2016.
3. Explain the making of timetables as a network problem.
   Elaborate an illustrative example.

4. simplex is not a polynomial-time algorithm

The title of this topic comes from the title of section 8.6 of the book *Combinatorial Optimization. Algorithms and Complexity* by Christos H. Papadimitriou and Kenneth Steiglitz, Dover 1998. Study also **How good is the simplex algorithm?** by Victor Klee and George J. Minty, in *Inequalities III*, pages 159-175, Academic Press, 1972.

1. Formulate the construction of the Klee-Minty cube.
2. Set up the formulation for any dimension to run experiments with the LP solver in GLPK.
3. Do you observe exponential running times for increasing dimensions?