ordinary differential equations

1. Problems and Examples
   - three problems
   - two examples
2. The Scientific Method
   - separation of variables
   - slowing down
   - cooling off
   - population modeling
3. Mechanical Models
   - a planar pendulum
   - a spring mass system

MCS 472 Lecture 26
Industrial Math & Computation
Jan Verschelde, 13 March 2026

ordinary differential equations

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Problem 1: slowing down

An attack submarine is cruising at 40 knots at a depth of 1000 feet when suddenly the reactor scrams.

After 1 minute way has dropped to 30 knots.

How long does the crew have to make repairs
before forward motion falls below steerageway of 2 knots?

Problem 2: cooling off

A house furnace fails on a cold winter's evening
when the outside (ambient) temperature of 20 degrees Fahrenheit.

Although initially at 70 degree Fahrenheit,
the inside temperature has fallen to 65 degrees Fahrenheit after 1 hour.

How long before the inside temperature reaches
the damaging temperature of 32 degrees Fahrenheit?

Problem 3: population modeling

Predict the future population of a developed country.

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Example 1: Planar Pendulum

A mass m is suspended on a rod of length L.

The motion of the pendulum stays in the plane,
so we have only one parameter: the angle,
measuring deviation from the stable equilibrium position.

Example 2: Sliding Mass Spring

The mass m is sliding along a frictionless horizontal table restrained by a spring.

The spring has constant k and we assume linear motion.

We are interested in the displacement $x(t)$,
measured from the equilibrium, in function of time t.

ordinary differential equations

separation of variables

The solution to the Ordinary Differential Equation (ODE)

$$\frac{dy}{dx} = f(x)g(y), \quad y(x_0) = y_0$$

has a unique solution.

We compute the solution via the separation of variables method.

an example

Consider:

$$\frac{dy}{dx} = \frac{x}{y}, \quad y(3) = 5.$$

In applying `SymPy.jl`, we first formulate the problem:

```
x = Sym("x")
y = SymFunction("y")
ode = Eq(diff(y(x), x), x/y(x))

   d            x
   --(y(x)) = ----
   dx         y(x)
```

calling `dsolve`

To solve an ODE with `SymPy`, we call `dsolve`:

```
sol = dsolve(ode)
```

returns

```
 2-element Vector{Sym}:
    Eq(y(x), -sqrt(C1 + x^2)),
     Eq(y(x), sqrt(C1 + x^2))
```

or equivalently $y(x) = \pm\sqrt{C_1 + x^2}$.

The parameter C_1 depends on the initial condition.

imposing the initial condition

The condition $y(3) = 5$ must be defined by a dictionary:

```
initcond = Dict(y(3) => 5)
```

Then we apply `dsolve` again on the ODE:

```
solivp = dsolve(ode, ics=initcond)
```

```
               ________
              /  2
y(x) = \/   x   + 16
```

and now we see that $y(x) = \sqrt{x^2 + 16}$ is the unique solution.

the scientific method

The scientific method consists in four steps:

1. Model the problem into an ODE.
2. Impose the data.
3. Solve the ODE.
4. Interpret the results.

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slowing down

An attack submarine is cruising at 40 knots at a depth of 1000 feet when suddenly the reactor scrams. After 1 minute way has dropped to 30 knots. How long does the crew have to make repairs before forward motion falls below steerageway of 2 knots?

Newton's rule equates the sum of the inertial forces to the sum of the external forces:

- The inertial force is $F = ma$, where m is the mass and a is the acceleration.
- The external force is the water resistance $R = -kv^2$, proportional to the square of the velocity, for some constant k.

So our model is

$$m \frac{d}{dt} v(t) = -k(v(t))^2,$$

where $v(t)$ is the velocity in function of time t.

formulating the ODE

```
m, k, t = Sym("m, k, t")
v = SymFunction(v)
ode = Eq(m*diff(v(t),t), -k*v(t)^2)
```

Let lump the constants m and k into one constant, dividing by m and replacing k/m by K.

```
lefteq = ode.lhs()/m
K = Sym("K")
righteq = subs(ode.rhs()/m, k/m=>K)
newode = Eq(lefteq, righteq)
```

```
   d                2
   --(v(t)) = -K*v (t)
   dt
```

solving the ODE

Recall: *submarine is cruising at 40 knots*
so the initial condition is $v(0) = 40$.

```
initcond = Dict(v(0)=>40)
sol = dsolve(newode, ics=initcond)
```

```
                 1
    v(t) = ----------
           K*t + 1/40
```

What is K?

imposing the data

Recall: *after one minute way has dropped to 30 knots*,
so we determine K using $v(1) = 30$.

```
Kequ = subs(sol, t=>1, v(1) => 30)
Ksol = solve(Kequ, K)
```

The `solve` returns `1-element Vector{Sym}`
so we use `Ksol[1]` in the next substitution:

```
vsol = subs(sol, K=>Ksol[1])
```

```
             1
v(t) = ---------
        t    1
       --- + --
       120   40
```

interpreting the results

The original equation was
how long before the steerageway fall below 2 knots?

```
vequ = subs(vsol, v(t) => 2)
solve(vequ, t)
```

which returns 57.

At 57 minutes the speed has dropped to 2 knots.

So the crew has 56 minutes left.

rowboat resistance

Exercise 1:

A coasting rowboat experiences

- subsurface resistance proportional to v^2, and
- bow-wave surface resistance proportional to v^3.

Model the velocity v of this craft.

Will it ever come to rest?

ordinary differential equations

cooling off

A house furnace fails on a cold winter's evening when the outside (ambient) temperature of 20 degrees Fahrenheit. Although initially at 70 degree Fahrenheit, the inside temperature has fallen to 65 degrees Fahrenheit after 1 hour. How long before the inside temperature reaches the damaging temperature of 32 degrees Fahrenheit?

Newton's cooling law states that the rate at which the temperature changes is proportional to the gradient driving out the heat.

Let $T(t)$ be the temperature at time t.

$$\frac{d}{dt}T(t) = k(T(t) - 20), \quad \text{for some constant } k.$$

The left side of the ODE is the change in the temperature.
The right side is the difference between the inside temperature and the outside temperature of 20 degree Fahrenheit.

formulating and solving the ODE

```
T = SymFunction("T")
ode = Eq(diff(T(t),t), k*(T(t) - 20))
```

The *initially at 70 degrees Fahrenheit* gives the initial condition:

```
initcond = Dict(T(0) => 70)
sol = dsolve(ode, ics=initcond)

                  k*t
    T(t) = 50*e        + 20
```

What is k?

imposing the data

Recall: *the inside temperature has fallen to 65 degrees Fahrenheit, after 1 hour.*

```
kequ = subs(sol, t=>1, T(1)=>65)
kval = solve(kequ, k)
```

which gives $\log(9/10)$, stored in an array.

```
Tsol = subs(sol, k=>kval[1])
```

```
                t*log(9/10)
    T(t) = 50*e            + 20
```

interpreting the results

How long before the inside temperature reaches the damaging temperature of 32 degrees Fahrenheit?

```
Tequ = subs(Tsol, T(t)=>32)
tval = solve(Tequ, t)
```

```
   1-element Vector{Sym}:
     log((6/25)^(1/log(9/10)))
```

To interpret this exact result,
we use a 64-bit floating-point approximation:

```
Float64(tval[1])
```

```
   13.545077553292497
```

It takes about 13 and a half hour before it starts freezing inside.

warming up

Exercise 2:

A can of pop at 34°F is removed from a refrigerator,
placed on a counter for awhile in a room at temperature of 75°F.

After 2 minutes the temperature of the can has risen to 35°F.

Predict the future evolution of the temperature of the can.

When will the temperature of the can reach 70°F?

ordinary differential equations

population modeling

Predict the future population of a developed country.

Considering the births,
the growth of the population is proportional to its size.

Let $P(t)$ denote the size of the population as function of time t.

$$\frac{d}{d\,t}P(t) = kP(t), \quad \text{for some constant } k.$$

formulating the ODE and adjusting the model

```
P = SymFunction("P")
ode = Eq(diff(P(t),t), k*P(t))
dsolve(ode)
```

```
                 k*t
    P(t) = C1*e
```

The exponential is not realistic. Deaths occur.

For a better model, consider the correction term,
similar to the second term in a Taylor series.
This second term is proportional to the square of the population size.

```
logistics = Eq(diff(P(t),t), k*P(t) - K*(P(t))^2)
```

This model is called the *logistics equation*.

solving the logistics equation

```
Psol = dsolve(logistics)

                  k*(C1 + t)
               k*e
   P(t) = -------------------
              / k*(C1 + t)     \
           K*\e              - 1/
```

To interpret this model, let us normalize the size of the population at the beginning of time to one.

```
initcond = Dict(P(0)=>1)
Psol1 = dsolve(logistics, ics=initcond)
```

simplifying the result

```
               /       /  K  \ \
               |    log|-----| |
               |       \K - k/ |
             k*|t + ---------- |
               \        k      /
          k*e
P(t) = ---------------------------
         /   /       /  K  \ \    \
         |   |    log|-----| |    |
         |   |       \K - k/ |    |
         | k*|t + ---------- |    |
         |   \        k      /    |
       K*\e                   - 1/
```

With `simplify()`, we obtain $P(t) = \dfrac{ke^{kt}}{Ke^{kt} - K + k}$.

interpreting the result

We obtained the expression

$$P(t) = \frac{ke^{kt}}{Ke^{kt} - K + k}.$$

In this expression, the two constants k and K are both positive.

If we divide numerator and denominator by e^{kt}:

$$P(t) = \frac{k}{K + (k - K)e^{-kt}}.$$

As $t \rightarrow \infty$, $P(t) \rightarrow k/K$.

This limit is called the *carrying capacity* of a society.

logistic model for the U.S. population

Proposal for Topic of a Project:

How well does the logistic model fit the population of the U.S.A.?

- Gather census data of the U.S. population since 1945.
- Fit the constants k and K.
- What is the carrying capacity? Interpret the results.

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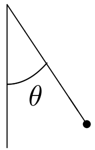
a planar pendulum

A mass m is suspended on a rod of length L.
The motion of the pendulum stays in the plane,
so we have only one parameter: the angle $\theta(t)$,
measuring deviation from the stable equilibrium position.



- Newton: tangential inertial force keeps the mass in place.
  The vertical component of the gravitational force is $-mg\sin(\theta(t))$.

$$F = mL\frac{d^2}{d\,t}\theta(t) = -mg\sin(\theta(t)).$$

- Hamilton: kinetic and potential energy remain constant.
  The kinetic energy is $\frac{1}{2}mL^2\left(\frac{d}{d\,t}\theta(t)\right)^2$. The gravitational force determines the potential energy is $mg(L - L\cos(\theta(t)))$.

sum of kinetic and potential energy

Conservation of energy is expressed as

$$\frac{1}{2}mL^2\left(\frac{d}{d\,t}\theta(t)\right)^2 + mg(L - L\cos(\theta(t))) = E,$$

for some constant E.

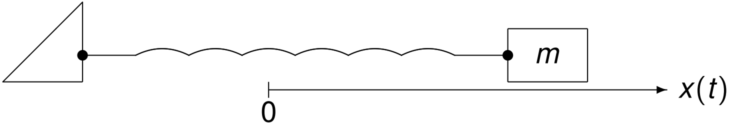
Exercise 3:
Show that the above equation is equivalent to the second order differential equation derived with Newton's rule.

Hint: apply $\dfrac{d}{d\,t}$ to the equation and factor.

ordinary differential equations

a spring mass system

Consider a mass m sliding along a frictionless horizontal table, restrained by a spring. The displacement from equilibrium is $x(t)$.



By Newton's law, the inertial force is $m\frac{d^2}{dt}x(t)$.

The restoring force of the spring is proportional to $x(t)$, so we obtain

$$m\frac{d^2}{dt}x(t) = -kx(t), \quad \text{for some constant } k.$$

summary and bibliography

With symbolic computation we applied the scientific method (formulate equations, impose data, solve, and interpret results) to several problems.

We covered the first two sections of Chapter 9 of our text book.

- Charles R. MacCluer:
  Industrial Mathematics. Modeling in Industry, Science, and Government. Prentice Hall, 2000.