Least Squares Chebyshev and Fourier

MCS 472 Lecture 17
Industrial Math & Computation
Jan Verschelde, 20 February 2026

Least Squares Chebyshev and Fourier

approximating functions with orthogonal bases

Consider an interval $[a, b]$ and a basis of functions ϕ_i, $i = 0, 1, 2, \ldots$, equipped with an inner product:

$$\langle f, g \rangle,$$

for any pair of functions f and g.

If for the basis functions ϕ_i we have

$$\langle \phi_i, \phi_j \rangle = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}$$

then we say that the basis $\{\phi_0, \phi_1, \phi_2, \ldots\}$ is *orthogonal*.

the least squares approximation

For any function $f(x)$ over $[a, b]$, consider the series

$$f(x) = \sum_{i=0}^{\infty} c_i \phi_i(x), \quad c_i = \langle f, \phi_i \rangle.$$

Truncate the series to the first $n + 1$ terms:

$$f(x) = p(x) + \sum_{i=n+1}^{\infty} c_i \phi_i(x), \quad \text{with} \quad p(x) = \sum_{i=0}^{n} c_i \phi_i(x).$$

Because $c_i = \langle f, \phi_i \rangle$ and ϕ_i is an orthogonal basis:

$$\langle f - p, \phi_i \rangle = 0, \quad i = 0, 1, \ldots, n,$$

and therefore p is *the least squares approximation* for f.

Least Squares Chebyshev and Fourier

the Chebyshev polynomials

Definition (the Chebyshev polynomial)

The *Chebyshev polynomial* $T_n(x)$ is $\cos(n \arccos(x))$.

Chebyshev polynomials can be computed via the recursion:

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x).$$

Verify: $T_0(x) = \cos(0) = 1$, $T_1(x) = \cos(\arccos(x)) = x$.

Consider the identities:

$$\begin{array}{rcrcl} \cos(a+b) & = & \cos(a)\cos(b) & + & \sin(a)\sin(b) \\ + \quad \cos(a-b) & = & \cos(a)\cos(b) & - & \sin(a)\sin(b) \\ \hline \cos(a+b) + \cos(a-b) & = & 2\cos(a)\cos(b) & & \end{array}$$

Apply the above identity to $a = n \arccos(x)$, $b = \arccos(x)$.

the Chebyshev basis

Any polynomial can be written in the Chebyshev polynomial basis.
Consider $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$, $T_0(x) = 1$, $T_1(x) = x$.

$$\begin{aligned}
T_0(x) &= 1 \\
T_1(x) &= x \\
T_2(x) &= 2x^2 - 1 \\
T_3(x) &= 4x^3 - 3x \\
T_4(x) &= 8x^4 - 8x^2 + 1 \\
T_5(x) &= 16x^5 - 20x^3 + 5x
\end{aligned}$$

$$\begin{bmatrix} T_0 \\ T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 & 0 \\ 0 & -3 & 0 & 4 & 0 & 0 \\ 1 & 0 & -8 & 0 & 8 & 0 \\ 0 & 5 & 0 & -20 & 0 & 16 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \end{bmatrix}$$

the Chebyshev basis is orthogonal

Consider the inner product

$$\langle f, g \rangle = \frac{2}{\pi} \int_{-1}^{+1} \frac{f(x)g(x)}{\sqrt{1-x^2}} dx.$$

For the Chebyshev polynomials T_i, we have

$$\langle T_i, T_j \rangle = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}$$

Therefore, the Chebyshev polynomials form an orthogonal basis.

the Chebyshev series

We can approximate any function f with a Chebyshev series:

$$f(x) = \frac{c_0}{2} + \sum_{i=1}^{\infty} c_i T_i(x) = p(x) + \sum_{i=n+1}^{\infty} c_i T_i(x), \quad \text{with } c_i = \langle f, T_i \rangle,$$

for $i = 0, 1, \ldots, n$, where

$$p(x) = \frac{c_0}{2} + \sum_{i=1}^{n} c_i T_i(x),$$

is the truncation of the Chebyshev series.

By construction $\langle f - p, T_i \rangle = 0$, $i = 0, 1, \ldots, n$.

The polynomial p is a least squares approximation for f.

Least Squares Chebyshev and Fourier

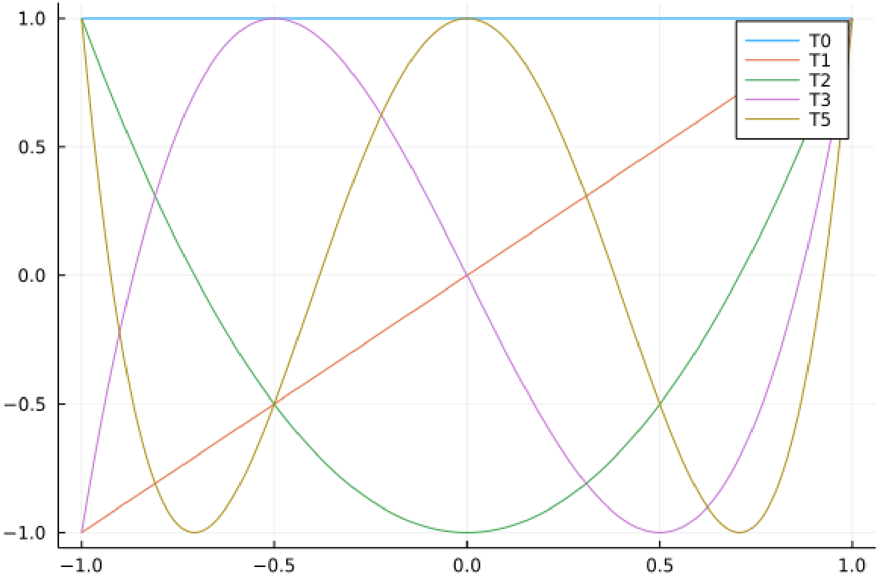
1. Approximating Functions
   - application of orthogonal bases
2. Chebyshev Approximations
   - the Chebyshev polynomials
   - computational experiments
3. Fitting Functions with Fourier Series
   - an orthogonal trigonometric basis
   - computational experiments
4. Proposals of Project Topics
   - fit average daily temperatures — currency fluctuations
   - on the origin of the Chebyshev polynomials

defining an array of basis functions

```
"""
   function basis(n::Int64)

returns an array of functions with the first n
Chebyshev polynomials.
"""
function basis(n::Int64)
   t0(x) = 1
   result = [t0]
   if(n > 1)
      t1(x) = x
      result = [result; t1]
      t1two(x) = 2*x
      for i=3:n
          newt(x) = t1two(x)*result[i-1](x) - result[i-2](x)
          result = [result; newt]
      end
    end
    return result
end
```

the first five Chebyshev polynomials



the inner product via Gauss-Chebyshev quadrature

$$\langle f, g \rangle = \frac{2}{\pi} \int_{-1}^{+1} \frac{f(x)g(x)}{\sqrt{1-x^2}} dx.$$

```
using FastGaussQuadrature

nodes, weights = gausschebyshev(16)
rule(f) = sum([weights[k]*f(nodes[k]) for k=1:length(weights)])

"""
    function innerproduct(f::Function, g::Function)

returns the value of the inner product of the functions f and g,
by the integral of f and g over [-1,+1] with weight 1/sqrt(1-x^2).
"""
function innerproduct(f::Function, g::Function)
   fun(x) = f(x)*g(x)
   return (2.0/pi)*rule(fun)
end
```

verifying the orthogonality of the basis

All inner products for the first five basis functions T_i, $i = 0, 1, 2, 3, 4$, at row i and column j is the value of $\langle T_i, T_j \rangle$:

```
+2.00e+00 +1.06e-16 -5.30e-17 +1.06e-16 -7.07e-17
+1.06e-16 +1.00e+00 +1.59e-16 -8.83e-17 +2.65e-16
-5.30e-17 +1.59e-16 +1.00e+00 +3.00e-16 -2.12e-16
+1.06e-16 -8.83e-17 +3.00e-16 +1.00e+00 +3.71e-16
-7.07e-17 +2.65e-16 -2.12e-16 +3.71e-16 +1.00e+00
```

We observe the identity matrix (modulo roundoff of machine precision), except for the first number in the first row and column, which equals 2.

the coefficients of the Chebyshev approximant

The function below computes $\langle f, T \rangle$ for all T in the basis.

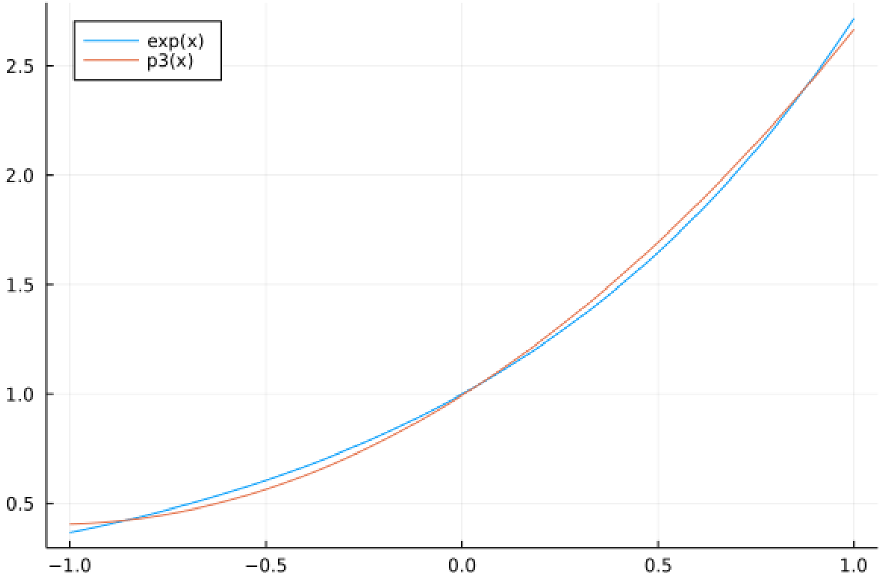
```
"""
    function coefficients(f::Function,b::Array{Function,1})

returns the inner products of f with the basis functions in b.
"""
function coefficients(f::Function,b::Array{Function,1})
   dim = length(b)
   result = zeros(dim)
   for i=1:dim
       result[i] = innerproduct(f,b[i])
   end
   return result
end
```

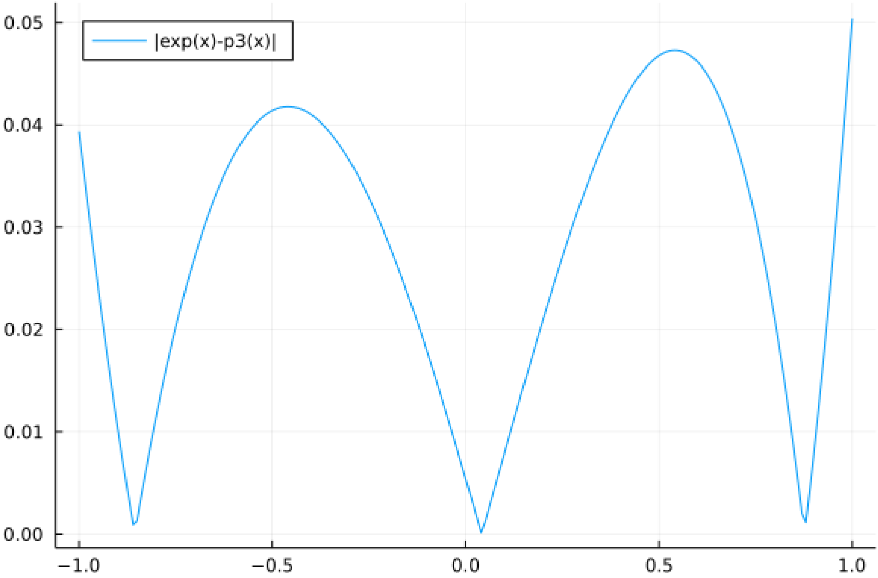
As an illustration, compute a cubic approximation for exp:

$$p(x) = \frac{c_0}{2} + \sum_{i=1}^{n} c_i T_i(x), \quad c_i = \langle \exp, T_i \rangle, \quad i = 0, 1, \ldots, n.$$

the cubic least squares approximant for exp(x)



the error of the cubic least squares approximant



two exercises

Exercise 1:
Consider the approximation of $\exp(x)$ by Chebyshev polynomials.

How large should the degree of the approximant be for the largest error to be less than 10^{-6} over $[-1, +1]$?

Exercise 2:
Consider $f(x) = 4\sin(2\pi x) + 3\sin(5\pi x)$ over $[-1, +1]$.

Compute a Chebyshev approximation with a high enough degree so the error is below 10^{-6} for all $x \in [-1, +1]$.

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an orthogonal trigonometric basis

Starting with the constant function 1,
we consider sine and cosine functions with increasing frequency,
respectively $\sin(k\pi t)$ and $\cos(k\pi t)$, for $k = 1, 2, \ldots, n$.

With respect to the inner product

$$\langle f, g \rangle = \int_{-1}^{+1} f(t)g(t)dt,$$

the functions in the basis are orthogonal to each other.

For any k and ℓ: $\langle \sin(k\pi t), \cos(\ell\pi t) \rangle = 0$ and

$$\langle \sin(k\pi t), \sin(\ell\pi t) \rangle = \begin{cases} 0, & k \neq \ell, \\ 1, & k = \ell. \end{cases}$$

The same holds for the cosine functions.

the least squares approximation

For any function $f(t)$ for $t \in [-1, +1]$:

$$p(t) = \frac{a_0}{2} + \sum_{k=1}^{n} \Big(a_k \cos(k\pi t) + b_k \sin(k\pi t) \Big),$$

where the coefficients are computed as

$$a_0 = \langle f, 1 \rangle, \quad a_k = \langle f, \cos(k\pi t) \rangle, \quad b_k = \langle f, \sin(k\pi t) \rangle.$$

Because of the orthogonal basis $f - p$ is orthogonal to the basis and therefore p is the least squares approximation for f.

For the interval $[0, 1]$, instead of $[-1, +1]$,
consider the basis functions $\cos(k2\pi t)$ and $\sin(k2\pi t)$, $k = 1, 2, \ldots$

Least Squares Chebyshev and Fourier

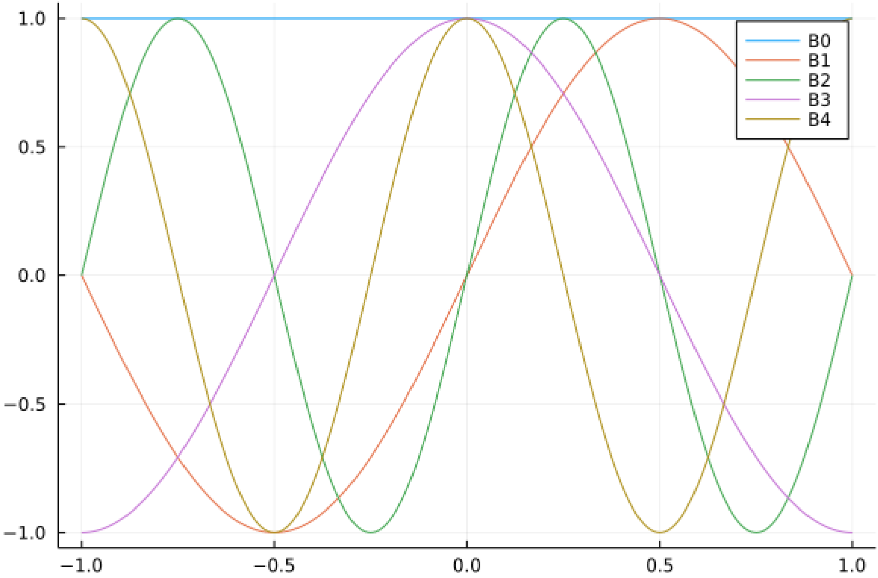
defining an array of basis functions

The basis functions are 1, $\cos(k\pi t)$, and $\sin(k\pi t)$, for $k = 1, 2, \ldots, n$.

```
"""
    function basis(n::Int64)

returns an array of functions with
the first 2*n+1 basis polynomials.
"""
function basis(n::Int64)
    one = [(t) -> 1]
    sines = [(t) -> sin(k*pi*t) for k=1:n]
    cosines = [(t) -> cos(k*pi*t) for k=1:n]
    result = [one; sines; cosines]
    return result
end
```

the first five basis functions



the inner product via Gauss-Legendre quadrature

$$\langle f, g \rangle = \int_{-1}^{+1} f(x)g(x)dx.$$

```
using FastGaussQuadrature


nodes, weights = gausslegendre(32)
rule(f) = sum([weights[k]*f(nodes[k]) for k=1:length(weights)])

"""
    function innerproduct(f::Function, g::Function)

returns the value of the inner product of the functions f and g,
by the integral of f and g over [-1,+1].
"""
function innerproduct(f::Function, g::Function)
   fun(x) = f(x)*g(x)
   return rule(fun)
end
```

verifying the orthogonality of the basis

All inner products for the first five basis functions B_i, $i = 0, 1, 2, 3, 4$, at row i and column j is the value of $\langle B_i, B_j \rangle$:

```
+2.00e+00 +6.26e-17 +5.57e-17 +4.46e-16 -1.04e-16
+6.26e-17 +1.00e+00 +1.93e-16 +2.79e-17 -1.78e-17
+5.57e-17 +1.93e-16 +1.00e+00 -3.37e-17 -5.71e-18
+4.46e-16 +2.79e-17 -3.37e-17 +1.00e+00 +2.93e-16
-1.04e-16 -1.78e-17 -5.71e-18 +2.93e-16 +1.00e+00
```

We observe the identity matrix (modulo roundoff of machine precision), except for the first number in the first row and column, which equals 2.

fitting a periodic function

Consider $f(t) = (\sin(2\pi t))^3$, for $t \in [-1, +1]$.

We compute the coefficients

$$a_0 = \langle f, 1 \rangle, \quad a_k = \langle f, \cos(k\pi t) \rangle, \quad b_k = \langle f, \sin(k\pi t) \rangle$$

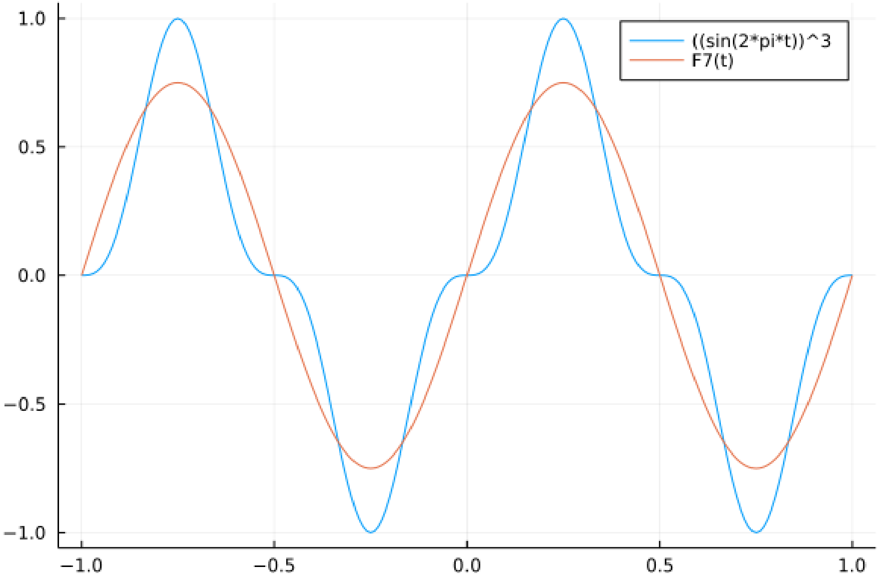
of

$$p(t) = \frac{a_0}{2} + \sum_{k=1}^{n} \Big(a_k \cos(k\pi t) + b_k \sin(k\pi t) \Big),$$

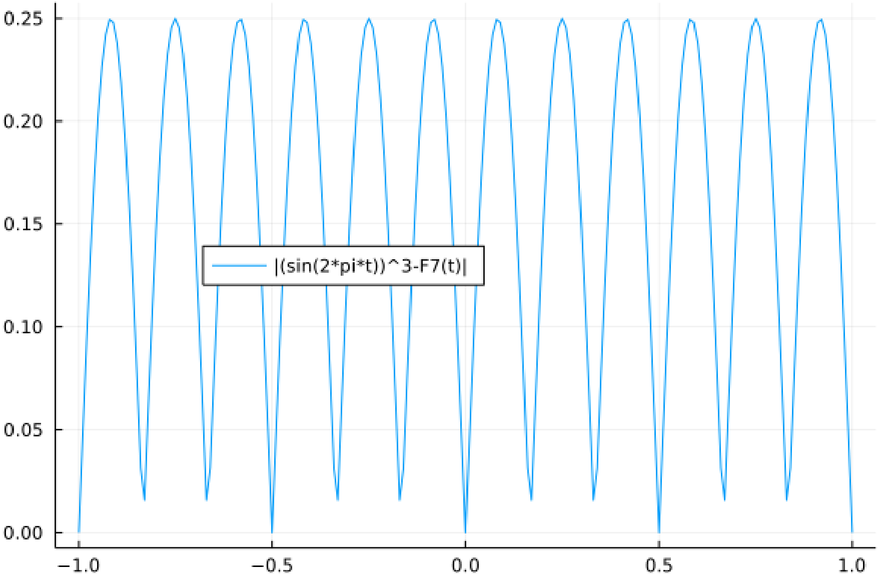
for

- $n = 3$ (7 basis functions) and
- $n = 6$ (13 basis functions).

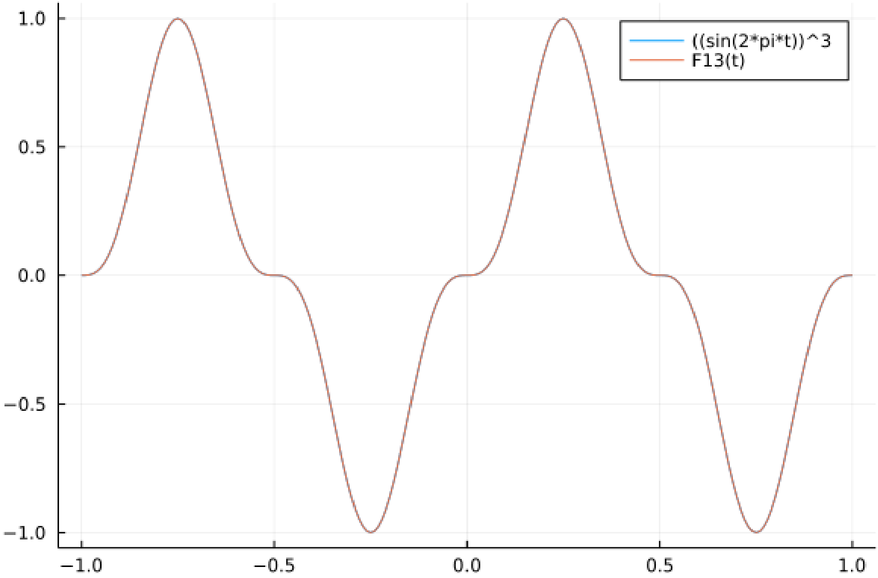
the approximation with 7 basis functions



the error with 7 basis functions



the approximation with 13 basis functions



looking at the Fourier series representation

For $n = 6$ (13 basis functions), only two coefficients are nonzero:

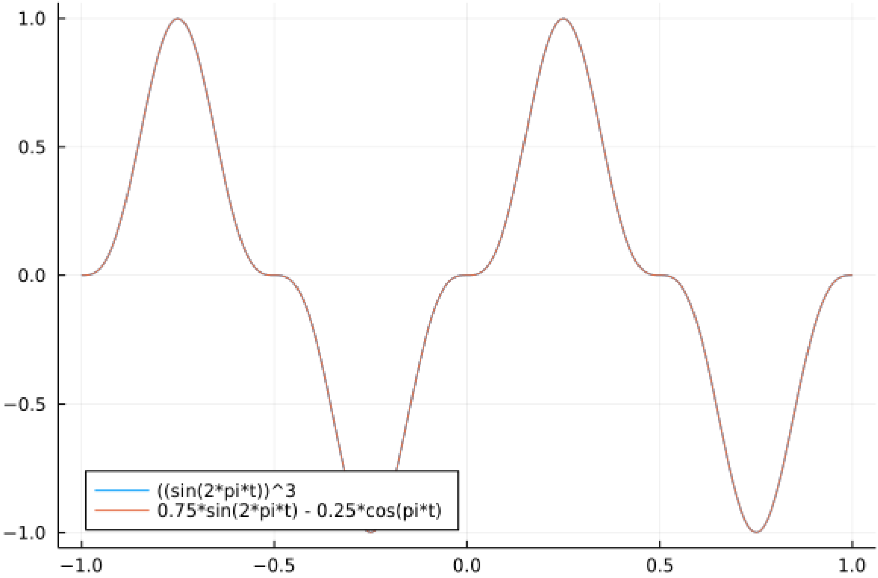
1. 0.75 at index 3, and
2. -0.25 at index 7.

These two indices respectively correspond to $\sin(2\pi t)$ and $\cos(\pi t)$.

Therefore, we can write

$$\left(\sin(2\pi t) \right)^3 = \frac{3}{4} \sin(2\pi t) - \frac{1}{4} \cos(\pi t).$$

the symbolic result



two exercises

Exercise 3:

Consider $f(x) = 4\sin(2\pi x) + 3\sin(5\pi x)$ over $[-1, +1]$.

- How many terms in the trigonometric basis are needed to represent f? Verify this numerically.
- Compare with the outcome of Exercise 2.
  Which method, Chebyshev polynomials or Fourier series, is preferred for this f.

Exercise 4:

Consider Fourier series approximations for $\exp(x)$ over $[-1, +1]$.

- How many terms in the Fourier series are needed for the largest error in the approximation to be less than 10^{-6}?
- Compare with the outcome of Exercise 1.
  Which method, Chebyshev polynomials or Fourier series, is preferred for $\exp(x)$?

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1. fit average daily temperatures

Obtain the average temperature for each day of one year.

Fit the data with a trigonometric basis.

Consider the following questions:

1. How large should the basis be for an accurate representation?
2. Compare the fit of one year on data of another year.
   How good does the fit predict the temperature of the other year?

2. fit USD/EUR currency fluctuations

Consider the evolution of the US dollar versus the Euro.

Which basis would be better? Chebyshev or Fourier?

Consider the following questions:

1. Gather data for at least one decade and fit the data with a Chebyshev and a Fourier basis. Which basis works best?
2. Is there a difference between the long term and short term? Compare one decade versus one month.

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3. on the origin of the Chebyshev polynomials

How did Chebyshev arrive at his polynomials?

Consider the paper by V. L. Goncharov:
The Theory of Best Approximation of Functions.
Journal of Approximation Theory 106:2–57, 2000.

The second section of this paper describes the memoir
Théorie des mécanismes connus sous le nom de parallélogrammes,
by Chebyshev, which appeared in 1854.

Consider the following questions:

1. Verify the reduction of computing a best approximating polynomial to a differential equation.
2. Illustrate your verification with some computational experiments.