The Delaunay Triangulation

1. Terrain Modeling
   - height interpolation

2. Triangulations of Planar Point Sets
   - definitions
   - complexity

3. Edge Flips
   - angle optimal triangulations
   - flipping edges
   - an algorithm for optimal angle triangulations

MCS 481 Lecture 26
Computational Geometry
Jan Verschelde, 18 March 2019
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terrain modeling

Given: a set $S$ of points in the plane, $S$ is a sample set, for every point we have a height measured.
Problem: represent the height at points outside $S$?

Solution: height interpolation using a triangulation of $S$.

1. Determine a triangulation of $S$ in the plane.
2. Map every triangle to 3-space, applying the heights to the points.
3. For every point inside a triangle, the height of the piecewise linear surface models the terrain.

Question: of the many possible triangulations, which one would be the best?
two triangulations

Consider two triangulations with labeled heights below:

The point $q$ receives one of the following heights:

1. at the left, $q$ has height $5 = (0 + 10)/2$,
2. at the right, $q$ has height $950 = (900 + 1000)/2$.

At the left, $q$ lies in a narrow valley (which looks wrong), at the right, $q$ lies on a mountain ridge (which looks better).

*The optimal triangulation maximizes the minimum angle of the triangles.*
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### Definition (maximal planar subdivision)

A subdivision $S$ is a *maximal planar subdivision* such that adding one edge connecting two vertices of $S$ destroys the planarity of $S$.

What does destroying the planarity mean?
→ Adding an edge intersects at least one existing edge.

### Definition (triangulation of a point set)

Let $P$ be a set of points in the plane. A *triangulation of the point set $P$* is a maximal planar subdivision which has $P$ as its vertex set.

Observe how this differs from the convex hull problem:
- every point of the given set $P$ must be a vertex,
- the number of edges is maximal.
all faces are triangles

Proposition (all faces are triangles)

Let $P$ be the vertex set of a planar subdivision $S$. If every face of $S$ is a triangle, then $S$ is maximal.

Exercise 1: write a proof for the above proposition.
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Theorem (complexity of a triangulation of a point set)

Let $P$ be a set of $n$ points, not all on the same line. Let $k$ be the number of points on the boundary of the convex hull of $P$. Any triangulation of $P$ has

1. $2n - 2 - k$ triangles, and
2. $3n - 3 - k$ edges.

If $P$ has no inner points of its convex hull, then $k = n$ and then any triangulation of $P$ has

1. $n - 2$ triangles, and
2. $2n - 3$ edges.

We proved in L-8 (Theorem 3.1 in the book) that a simple polygon with $n$ vertices has a triangulation with $n - 2$ triangles.
Euler’s formula for a connected graph

**Theorem (Euler’s formula for a connected graph)**

Let \( n_f, n_e, n_v \) respectively be the number of faces, the number of edges, and the number of vertices in a connected graph:

\[
    n_f - n_e + n_v = 2.
\]

1. \( n_v = n \), the number of points in \( P \).
2. Let \( m \) be the number of triangles, then \( n_f = m + 1 \).

   With the \(+1\), we count the unbounded face of the triangulation.
3. \( n_e = (3m + k)/2 \), as we have \( m \) triangles and every triangle has 3 edges. The unbounded face has \( k \) edges (really?). Every edge is shared by exactly 2 triangles: \((3m + k)/2\).

By Euler’s formula:

\[
    n - (3m + k)/2 + m + 1 = 2
\]

\[
    \Rightarrow 2n - 3m - k + 2m + 2 = 4 \Rightarrow 2n - 2 - k = m.
\]

Then, \( n_e = (6n - 6 - 3k - k)/2 = 3n - 3 - k \). Q.E.D.
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the angle vector of a triangulation

Definition (the angle vector of a triangulation)

Let $T$ be a triangulation of the point set $P$, $m = \#T$. The \textit{angle vector of the triangulation} $T$ is $A(T) = (\alpha_1, \alpha_2, \ldots, \alpha_{3m})$, a vector of $3m$ angles, sorted in increasing order.

By the lexicographic order on vectors, we obtain an order on triangulations.

Let $T$ and $T'$ be two triangulations of the same point set $P$:

$$T > T' \iff A(T) >_{\text{lex}} A(T').$$

For two angle vectors $\alpha$ and $\beta$ of size $3m$:

$$\alpha > \beta \iff \alpha_j = \beta_j, \text{ for all } j < i \text{ and } \alpha_i > \beta_i, i \leq 3m.$$
Definition (angle optimum triangulation)

A triangulation $T$ of $P$ is an *angle optimum triangulation* if $T \geq T'$, for all triangulations $T'$ of $P$.

Exercise 2: Give an example of a point set for which there is more than one angle optimum triangulation.
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flipping edges

Consider a convex quadrilateral spanned by vertices $a$, $b$, $c$, and $d$. The quadrilateral has two triangulations:

1. $T_1$: the triangles spanned by $(a, c, d)$ and $(b, c, d)$, and
2. $T_2$: the triangles spanned by $(a, b, c)$ and $(a, b, d)$.

The two triangulations $T_1$ and $T_2$ are related as follows:

1. transform $T_1$ into $T_2$ by replacing $(c, d)$ by $(a, b)$,
2. transform $T_2$ into $T_1$ by replacing $(a, b)$ by $(c, d)$.

This kind of replacement is called an **edge flip**.
illegal edges

Consider the angles inside the triangles below.

\[ \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6 \]

**Definition (illegal edge)**

The edge \((p_i, p_j)\) is an *illegal edge* if \( \min_{i=1}^{6} \alpha_i < \min_{i=1}^{6} \beta_i \).
Let $\angle pqr$ denote the angle between
- the edge $(p, q)$ and
- the edge $(q, r)$.

**Theorem (Thales circle theorem)**

For the circle $C$ and a line, $C \cap \ell = \{a, b\}$, $p, q \in C$, $r$ inside $C$, $s$ outside $C$:

$\angle arb > \angle apb = \angle aqb > \angle asb$. 
application of the Thales circle theorem

If all points $p_i$, $p_j$, $p_k$, and $p_\ell$ are on the same circle, then both edges $(p_i, p_j)$ and $(p_k, p_\ell)$ are legal.

\[ \angle p_i p_\ell p_j > \angle p_i p_k p_j \implies \text{edge } (p_i, p_j) \text{ is illegal} \]

Exercise 3: Write pseudo code for a function which returns true if $(p_i, p_j)$ is illegal, false otherwise, given $p_i$, $p_j$, $p_k$, and $p_\ell$ on input.
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an algorithm for optimal angle triangulations

A *legal triangulation* contains no illegal edges.

**Algorithm LEGALTRIANGULATION(\(T\))**

Input: a triangulation \(T\) of a point set \(P\).
Output: a legal triangulation of \(P\).

1. while \(T\) contains an illegal edge \((p_i, p_j)\) do
2. let \((p_i, p_j, p_k)\) and \((p_i, p_j, p_\ell)\) be adjacent triangles to \((p_i, p_j)\)
3. replace the edge \((p_i, p_j)\) by the edge \((p_k, p_\ell)\)
4. return \(T\)

Why does LEGALTRIANGULATION terminate?
→ With each edge flip, the angle vector of \(T\) increases.
Definition (the Delaunay triangulation)

Given a point set $P$, the Delaunay triangulation of $P$ maximizes the minimal angle over all triangulations of $P$. 
recommended assignments

We covered section 9.1 in the textbook.

Consider the following activities, listed below.

1. Write the solutions to exercises 1, 2 and 3.
2. Consult the CGAL documentation and example code on Delaunay triangulations.
3. Consider the exercises 1, 2, 3 in the textbook.