Voronoi Diagrams

1. Definition
   - the post office problem
   - Voronoi diagrams with scipy
   - definition and basic properties
2. Complexity
   - number of vertices and edges
   - application of Euler's formula
3. Characterization of Vertices and Edges
   - the largest empty circle
   - proof of the characterizations

MCS 481 Lecture 21
Computational Geometry
Jan Verschelde, 3 March 2025

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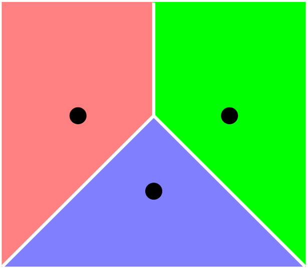
the post office problem

Consider the locations of three post offices,
marked by the black dots on the picture below:



People go to their closest post office:

- all points in regions of the *same* color have the *same* post office as their closest post office.
- points on the white lines are at the same distance of post offices.

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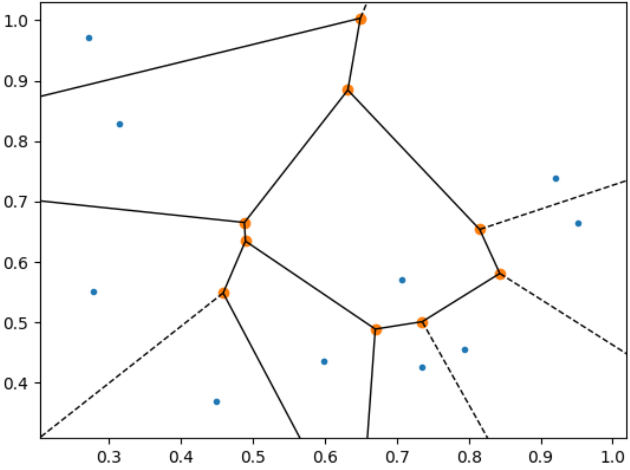
Voronoi diagrams with scipy

We use `numpy` to generate 10 random points.
The `spatial` package of `scipy` provides `Voronoi` to construct the Voronoi diagram and we plot with `voronoi_plot_2d` using `matplotlib`.

```
import numpy as np
from scipy.spatial import Voronoi, voronoi_plot_2d
from matplotlib import pyplot as plt

points = np.random.rand(10, 2)
vor = Voronoi(points)
voronoi_plot_2d(vor)
plt.show()
```

the Voronoi diagram of 10 random points



Voronoi Diagrams

definitions

Definition (distance between two points)

Let $p = (p_x, p_y)$ and $q = (q_x, q_y)$. The *distance between p and q* is

$$\mathrm{dist}(p, q) = \sqrt{(p_x - q_x)^2 + (p_y - q_y)^2}.$$

The input is a set of points, each point is called a *site*.

Definition (Voronoi cell and Voronoi diagram)

Let $P = \{p_1, p_2, \ldots, p_n\}$ be n points in the plane. The collection of all points closest to $p_i \in P$ is *the Voronoi cell $V_P(p_i)$ of p_i*:

$$V_P(p_i) = \{\ q \in \mathbb{R}^2 \mid \mathrm{dist}(q, p_i) < \mathrm{dist}(q, p_j),\ \text{for all } j \neq i\ \}.$$

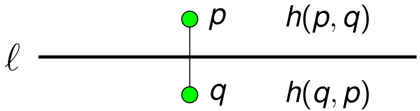
The *Voronoi diagram $Vor(P)$ of P* is

$$\mathrm{Vor}(P) = \{\ V_P(p_i), i = 1, 2, \ldots, n\ \}.$$

the bisector of two points

Definition (bisector of two points)

Let p and q be two points in the plane. The bisector ℓ of p and q is the perpendicular line through the middle of the line segment (p, q).

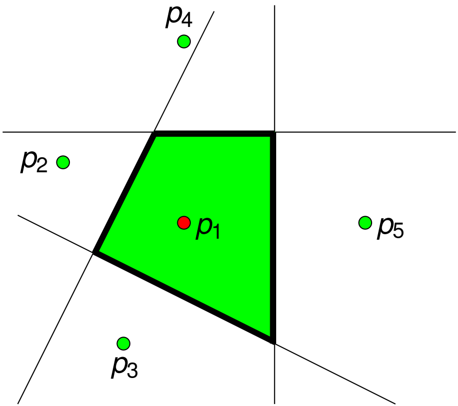


Definition (halves of two points)

Let p and q be two points in the plane. The bisector ℓ of p and q splits the plane in two halves. The halve $h(p, q)$ is the half plane which contains p. The halve $h(q, p)$ is the half plane which contains q.

Observe: $r \in h(p, q) \Leftrightarrow \text{dist}(r, p) < \text{dist}(r, q)$.

Voronoi cells are convex



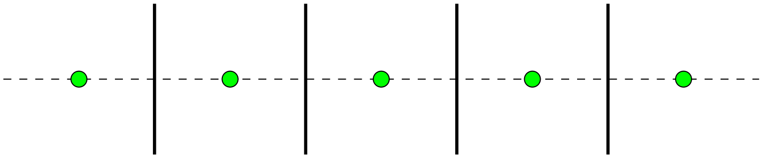
$$V_P(p_1) = h(p_1, p_2) \cap h(p_1, p_3) \cap h(p_1, p_4) \cap h(p_1, p_5)$$

edges of Voronoi diagrams

Theorem (edges of Voronoi diagrams)

If all n points in P are collinear, then $\mathrm{Vor}(P)$ consists of $n-1$ parallel lines. Otherwise, all edges of $\mathrm{Vor}(P)$ are either line segments or half lines. Moreover, all edges in $\mathrm{Vor}(P)$ are connected.

The case of collinear points gets a picture proof:



Where is the closest subway stop?

edges are either segments of half lines

If the points are not collinear, then edges are segments of half lines.

We proof this statement by contradiction.

Assume the edge $\ell \in V_P(p_i) \cap V_P(p_j)$ is a full line.

Consider p_k, $k \neq i$, $k \neq j$.

> p_k is not collinear with p_i and p_j
> $\Rightarrow$ The bisector of p_k and p_i is neither collinear nor parallel
> to the bisector of p_k and p_j.
> Either $\ell \in h(p_k, p_i)$ or $\ell \in h(p_k, p_j)$.

But this contradicts that ℓ is a full line.

the edges are connected

If the points are not collinear, then the edges are connected.

We proof this statement by contradiction.

Assume edges of Vor(P) are not connected.

$\Rightarrow$ Some cell $V_P(p_i)$ splits the plane in two.
$\Rightarrow$ $V_P(p_i)$ is bounded by two full, parallel lines.

But we can only full lines as edges if the points are collinear.

This leads to a contradiction and the assumption is false.

Q.E.D.

a first algorithm

Exercise 1: Consider $P = \{ (2,3), (0,4), (1,1), (2,6), (5,3) \}$.

1. List all vertices and edges of Vor(P).
2. A first algorithm for Vor(P) computes $V_P(p_i)$, for $i = 1, 2, \ldots, n$, via the bisectors of p_i with all other points.
   Describe this first algorithm using pseudo code.
3. What is the cost, as function of $n = \#P$, of this algorithm?

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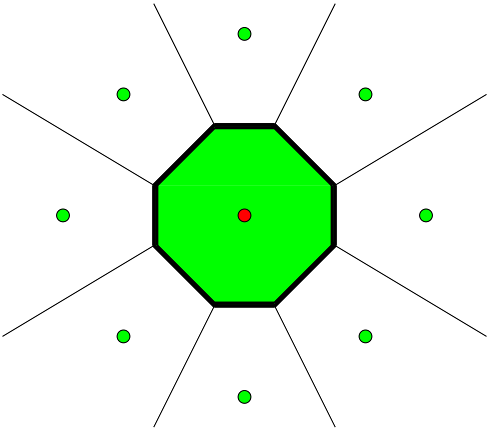
number of vertices and edges of a Voronoi cell

A Voronoi cell can have $n - 1$ vertices and $n - 1$ edges:



If P has n points, can the size of $\text{Vor}(P)$ be $O(n^2)$?

counting the number of vertices and edges

Exercise 2: Consider $P = \{ (10,35), (20,15), (20,55), (40,5), (40,35), (40,65), (60,15), (60,55), (70,35) \}$.

Count the number of vertices and edges in Vor(P).

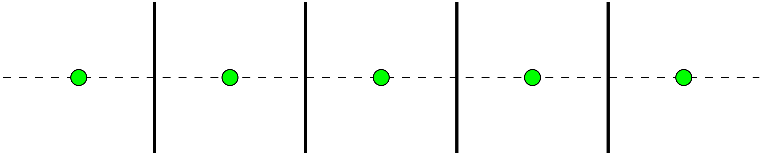
complexity of a Voronoi diagram

Theorem (complexity of a Voronoi diagram)

Let $\mathrm{Vor}(P)$ *be a Voronoi diagram for n points,* $n \geq 3$.

1. *The number of vertices of* $\mathrm{Vor}(P)$ *is at most* $2n - 5$.
2. *The number of edges of* $\mathrm{Vor}(P)$ *is at most* $3n - 6$.

Let us start with a collinear set P:



For n sites, we have $n - 1$ parallel lines and one single vertex at ∞.

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application of Euler's formula

Theorem (Euler's formula)

Let n_f, n_e, n_v respectively be the number of faces, the number of edges, and the number of vertices in a graph: $n_f - n_e + n_v \geq 2$.

To close the half line edges, we add v_∞ as the vertex at infinity.

For every site, we have one face, so $n_f = n$, $n = \#P \geq 3$.

Every edge has two vertices, the degree of each vertex is at least 3:

$$2n_e \geq 3(n_v + 1), \qquad \text{the } +1 \text{ stems from } v_\infty.$$

Apply Euler's formula: $n_f - n_e + (n_v + 1) \geq 2$ and eliminate n_e:

$$\begin{aligned} n = n_f &\geq 2 + n_e - (n_v + 1) \\ &\geq 2 + \frac{3}{2}(n_v + 1) - n_v - 1 = \frac{1}{2}n_v + \frac{5}{2} \Rightarrow n_v \leq 2n - 5. \end{aligned}$$

a bound on the number of edges

We start again with Euler's formula:

$$n_f - n_e + (n_v + 1) \geq 2 \quad \Rightarrow \quad n_e \leq n_f + (n_v + 1) - 2$$

and eliminate n_v with $3(n_v + 1) \leq 2n_e$:

$$3(n_v + 1) \leq 2n_2 \quad \Rightarrow \quad n_v \leq \frac{2}{3}n_e - 1.$$

The number of faces $n_f = n$, the number of sites:

$$\begin{aligned} n_e &\leq n + \frac{2}{3}n_e + 1 - 1 - 2 \\ \frac{1}{3}n_e &\leq n - 2 \Rightarrow n_e \leq 3n - 6. \end{aligned}$$

Q.E.D.

practical interpretation

For P not collinear, the number of bisectors is $O(n^2)$.
But Vor(P) does not store all bisectors.

Theorem (complexity of a Voronoi diagram)

Let $\mathrm{Vor}(P)$ be a Voronoi diagram for n points, $n \geq 3$.

1. *The number of vertices of $\mathrm{Vor}(P)$ is at most $2n - 5$.*
2. *The number of edges of $\mathrm{Vor}(P)$ is at most $3n - 6$.*

Even though some cells may have $n - 1$ edges and vertices,
the complexity of Vor(P) is $O(n)$.

This implies the possibility of an $O(n \log(n))$ algorithm.

Exercise 3: What statement can you make about the *average* number of vertices of a Voronoi diagram for n points?
Justify either experimentally or via the above theorem.

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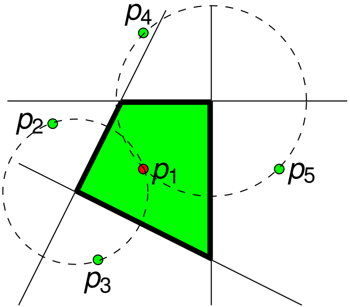
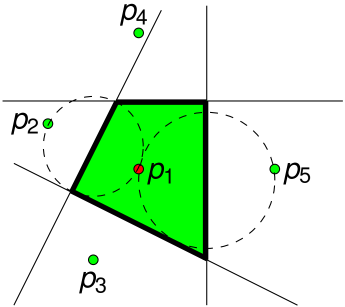
the largest empty circle

Which bisectors define edges? Which intersections are vertices?

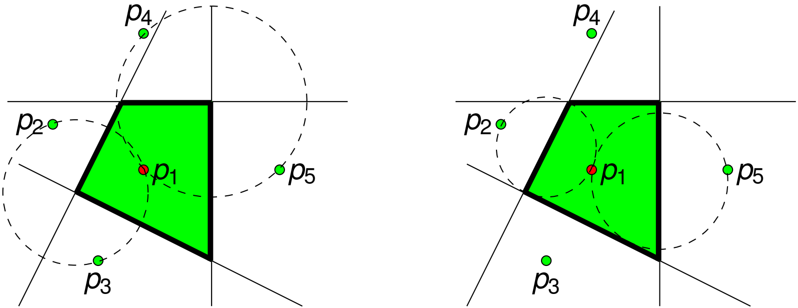
Definition (largest empty circle)

Let P be a set of points and some point q.
The *largest empty circle $C_P(q)$ of q with respect to P* is the largest circle centered at q that does not contain a point of P in its interior.





characterization of vertices and edges



Theorem (characterization of vertices and edges)

Let $\mathrm{Vor}(P)$ be the Voronoi diagram of the set P.

1. *q is a vertex of $\mathrm{Vor}(P)$*
   $\Leftrightarrow$ $C_P(q)$ contains 3 or more points of P on its boundary.
2. *e is an edge of $\mathrm{Vor}(P)$ $\Leftrightarrow$ e is on the bisector between p_i and p_j and there is a $q \in e$ so $C_P(q)$ has only p_i and p_j on its boundary.*

Voronoi Diagrams

characterization of a vertex, part one

Lemma (characterization of a vertex, part one)

q is a vertex of $\mathrm{Vor}(P)$
$\Rightarrow C_P(q)$ *contains 3 or more points of P on its boundary.*

If q is a vertex, then it is incident to at least 3 sites p_i, p_j, and p_k and their Voronoi cells $V(p_i)$, $V(p_j)$, and $V(p_k)$.

This implies that q is equidistant to p_i, p_j, and p_k.
Moreover, there is no other site closer than p_i, p_j, and p_k.

Three points uniquely define the circle $C_P(q)$.

characterization of a vertex, part two

Lemma (characterization of a vertex, part two)

q is a vertex of $\mathrm{Vor}(P)$
$\Leftarrow C_P(q)$ *contains 3 or more points of P on its boundary.*

If $C_P(q)$ has three points p_i, p_j, p_k of P on its boundary
and it contains no other points of P in its interior,
then the three bisectors:

1. between p_i and p_j,
2. between p_i and p_k, and
3. between p_j and p_k

all pass through q.

This implies that q is a vertex of $\mathrm{Vor}(P)$.

characterization of an edge, part one

Lemma (characterization of an edge, part one)

e is an edge of $\mathrm{Vor}(P) \Rightarrow e$ is on the bisector between p_i and p_j and there is a $q \in e$ so $C_P(q)$ has only p_i and p_j on its boundary.

- If e is an edge of Vor(P), then e lies on the bisector between two points p_i and p_j.
- Every point on the bisector is equidistant to p_i and p_j.
- Take the midpoint of the edge as the center of the circle with radius the distance from the midpoint to p_i and p_j.
- Take q as that center, which gives the circle $C_P(q)$, which has only p_i and p_j on its boundary.

characterization of an edge, part two

Lemma (characterization of an edge, part two)

e is an edge of $\mathrm{Vor}(P) \Leftarrow e$ is on the bisector between p_i and p_j and there is a $q \in e$ so $C_P(q)$ has only p_i and p_j on its boundary.

If e is on the bisector between p_i and p_j, which are the only points on the boundary of $C_P(q)$, for $q \in e$, then

$$\mathrm{dist}(q, p_i) = \mathrm{dist}(q, p_j) < \mathrm{dist}(q, p_k), \text{ for all } k \neq i, k \neq j.$$

This implies that q is a vertex or lies on an edge.
But we need 3 points for q to be a vertex.

Therefore, q lies on an edge defined by a bisector.

Q.E.D.

summary and exercises

Defining Voronoi diagrams, we covered section 7.1 in the textbook.

Consider the following activities, listed below.

1. Write the solutions to exercises 1, 2, and 3.
2. Consult the CGAL documentation and example code on Voronoi diagrams.
3. Consider the exercises 1, 2, 3 in the textbook.