The Zone Theorem

1. An Example of the CGAL
   - finding collinear points via duality
2. Arrangements of Lines
   - an incremental algorithm
   - the zone of a line and the zone theorem
3. Levels and Discrepancy
   - counting the number of lines
   - computing the level at vertices
4. Proof of the Zone Theorem
   - induction on the number of lines

MCS 481 Lecture 25
Computational Geometry
Jan Verschelde, 18 October 2023

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finding collinear points via duality

In `examples/Arrangements_on_surface_2` of the software CGAL, the program `dual_lines.cpp` illustrates an application of arrangements of lines.

Given a set of points, does the set contain three collinear points?

The points p_1, p_2, p_3 lie on the line ℓ
$\Leftrightarrow$ the point ℓ^* lies on the lines p_1^*, p_2^*, p_3^*.

The file `points.dat` contains 100 points, no collinear ones.

Exercise 1: Run the code on a point set with collinear points.

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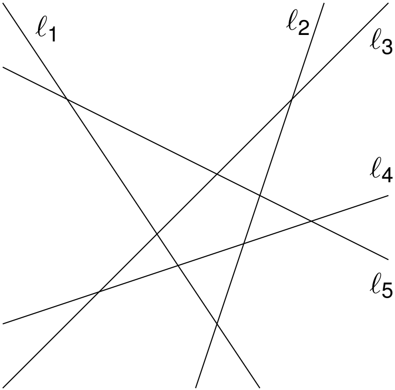
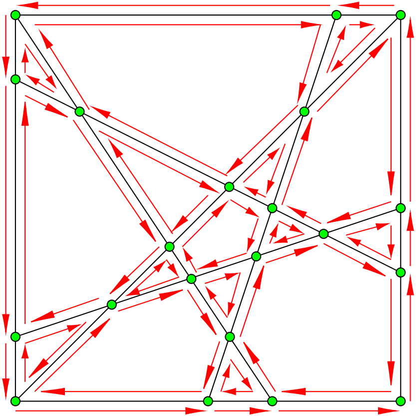
five lines in the plane induce a subdivision





The arrangement of lines is stored in a doubly connected edge list, within a bounding box.

an incremental algorithm

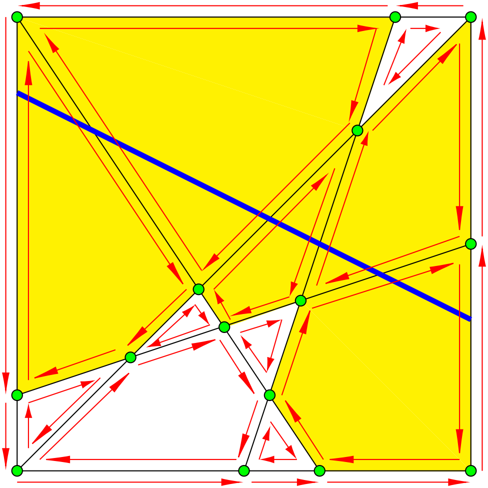
Algorithm CONSTRUCTARRANGEMENT(L)

Input: a set L of n lines.
Output: $A(L)$, stored in doubly connected edge list,
within a bounding box $B(L)$.

compute $B(L)$ enclosing all vertices of $A(L)$
construct a doubly connected edge list $\mathcal{D}$ to store $B(L)$
for i from 1 to n do
    find the edge e on $B(L)$
    that contains leftmost intersection point of ℓ_i and A_{i-1}
    let f be the bounded face incident to e
    while f is not outside $B(L)$ do
        split f, update $\mathcal{D}$
        set f to the next face intersected by ℓ_i

The Zone Theorem

the zone of a line – an example



The blue line intersects 5 of the 11 faces.

the zone of a line – definitions

Definition (the closure of a face in a subdivision)

Let f be a face in a subdivision.
The *closure $\bar{f}$ of the face f* is f and all its vertices and edges.

Definition (the zone of a line in an arrangement)

Let $A(L)$ be a line arrangement and ℓ be a line.
The *zone of the line ℓ in the arrangement $A(L)$* is

$$\{ \, f \text{ face of } A(L) \mid \bar{f} \cap \ell \neq \emptyset \, \}.$$

Definition (the zone complexity)

Let $A(L)$ be a line arrangment and ℓ be a line.
The *zone complexity of ℓ in $A(L)$* is the sum of the number of vertices, the number of edges, and the number of faces in the zone of ℓ in $A(L)$.

the zone theorem

Theorem (the zone theorem)

Let L be a set of m lines and ℓ be some line.
The zone complexity of a line ℓ in the arrangement $A(L)$ is $O(m)$.

By the zone theorem,
the cost of the incremental algorithm is quadratic.

Theorem (cost of CONSTRUCTARRANGEMENT)

A doubly connected edge list for the arrangement induced by a set of n lines in the plane can be constructed in $O(n^2)$ time.

The Zone Theorem

dual lines and dual points

Definition (dual lines and dual points)

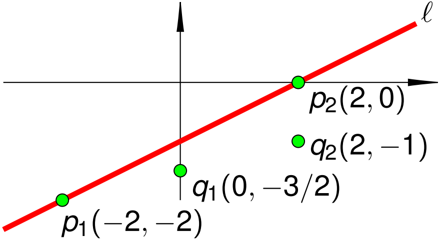
Given a point p with coordinates (p_x, p_y),
the dual p^* of the point p is the line $y = p_x x - p_y$.
Given the line ℓ with slope m and intercept b, $y = mx + b$,
the dual ℓ^* of the line ℓ is the point with coordinates $(m, -b)$.

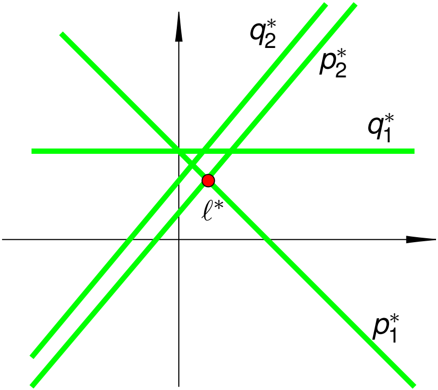
$$
\begin{array}{rcl}
p_1 : (-2, \; -2 \;) & \Leftrightarrow & p_1^* : y = -2x + 2 \\
p_2 : (\; 2, \; 0 \;) & \Leftrightarrow & p_2^* : y = \; 2x \\
q_1 : (\; 0, \; -3/2 \;) & \Leftrightarrow & q_1^* : y = \; + 3/2 \\
q_2 : (\; 2, \; -1 \;) & \Leftrightarrow & q_2^* : y = \; 2x + 1 \\
\ell : y = (1/2)x - 1 & \Leftrightarrow & \ell^* : (\; 1/2, \; +1 \;) \\
q_1 \text{ is below } \ell & \Leftrightarrow & \ell^* \text{ is below } q_1^*
\end{array}
$$

the dual of the discrepancy problem

Given a line ℓ, we want to count all points below ℓ.

$\ell : y = (1/2)x - 1$
$\ell^* : (1/2, 1), q_1^* : y = 3/2$
$p_2^* : y = 2x, q_2^* : y = 2x + 1$
$p_1^* : y = -2x + 2$



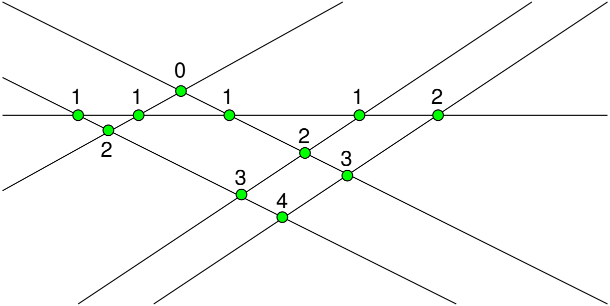


Given the point ℓ^*, count the lines above ℓ^*.

the level of a point in an arrangement

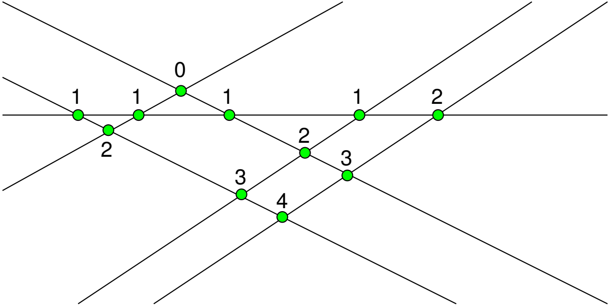
Definition (level of a point in an arrangement)

Given an arrangement $A(L)$ of lines and a point p,
the *level of the point p in $A(L)$* is the number of lines strictly above p.



The Zone Theorem

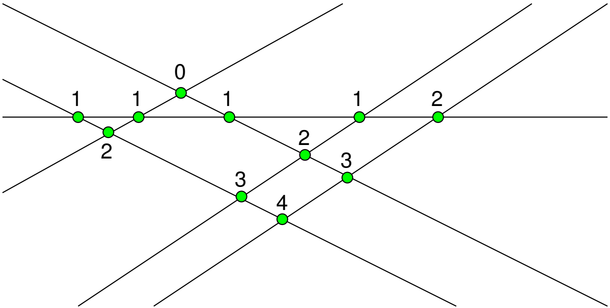
walking a line and computing levels



For any line ℓ do the following:

1. compute the level at the leftmost vertex,
2. while not at the rightmost vertex on ℓ do
3. walk to the next vertex v on ℓ and compute the level of v.

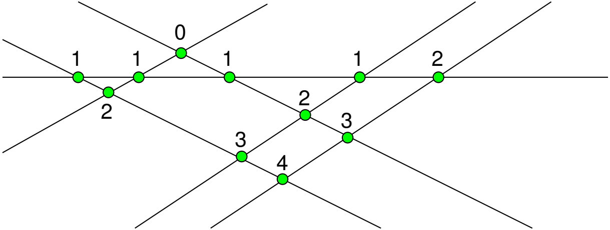
computing the level of the leftmost vertex



In an arrangement of n lines, and a given line ℓ,
computing the level of the leftmost vertex on ℓ runs in $O(n)$.

$\rightarrow$ for the vertex v on ℓ with the smallest x-coordinate,
check all other $n - 1$ lines to see whether v lies below.

computing the level of the next vertex



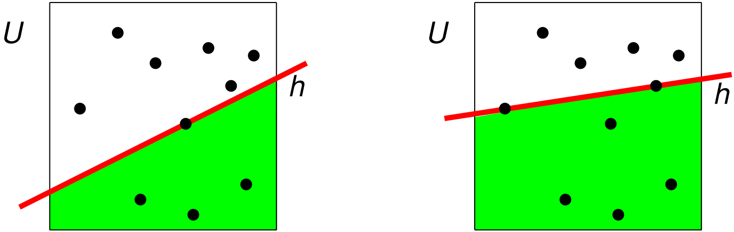
In an arrangement of n lines, and a given line ℓ,
computing the level of the next vertex on ℓ also runs in $O(n)$.

$\rightarrow$ in the walk from one vertex to the next, we follow the edges in the doubly connected edge list and update the level as follows:

$+1$ if the edge we follow goes down,
-1 if the edge we follow goes up.

computing the discrete measure in quadratic time

The discrete measure of S in U is $\mu_S(h) = \#(S \cap U)/\#S$.



The dual of the sample set S of points is the set of lines S^*.
We count the levels of the vertices in the arrangement $A(S^*)$.

Theorem (cost of half plane discrepancy)

The half plane discrepancy of a set S of n points in the unit square U can be computed in $O(n^2)$ time.

The Zone Theorem

proving the zone theorem

Theorem (the zone theorem)

Let L be a set of m lines and ℓ be some line.
The zone complexity of a line ℓ in the arrangement $A(L)$ is $O(m)$.

Outline of the proof:

- Choose the coordinate system so that ℓ is the x-axis.
- Each edge in $A(L)$ bounds two faces.
  An edge is a *left bounding edge* for the face to its *right*.
  An edge is a *right bounding edge* for the face to its *left*.
- In the zone of ℓ, the number of left bounding edges $\leq 5m$.

The theorem follows from the last statement.

the number of left bounding edges

Lemma (the number of left bounding edges)

Let L be a set of m lines and ℓ be the x-axis.
In the zone of ℓ in $A(L)$, the number of left bounding edges $\leq 5m$.

The lemma is proven by induction on m.

- The base case: $m = 1$, only one line in L,
  5 is indeed an upper bound to the number of left bounding edges.

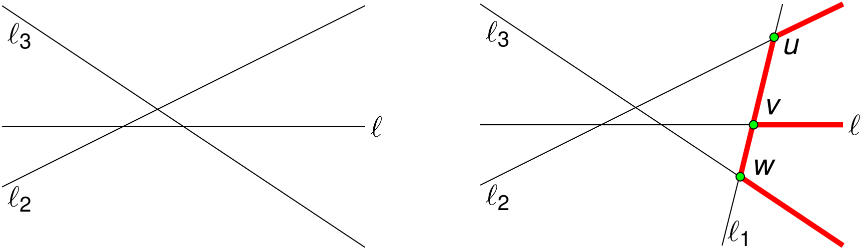
- The general case.
  Let ℓ_1 be the line in L that has the rightmost intersection with ℓ.

  We apply the induction hypothesis to $A(L \setminus \{\ell_1\})$:
  in $A(L \setminus \{\ell_1\})$, the number of left bounding edges $\leq 5(m-1)$.

  Need to show:
  no more than 5 new left bounding edges when ℓ_1 is added.

the general case

We first assume ℓ_1 intersects ℓ only at one point v:

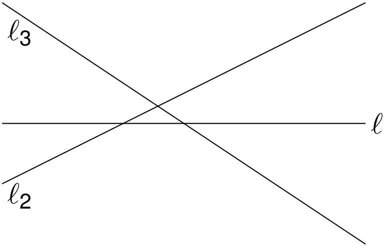


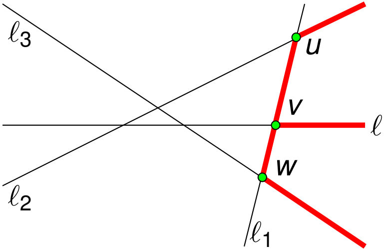
We see 5 new edges.

1. The edge on ℓ_1, spanned by (u, v).
2. The edge on ℓ_1, spanned by (v, w).
3. The edge on ℓ_2, starting at u.
4. The edge on ℓ, starting at v.
5. The edge on ℓ_3, starting at w.

the general case – continued

We first assume ℓ_1 intersects ℓ only at one point v:





The 5 new edges may not the only new edges.
However, other new edges are above the vertex u or below w and therefore do not belong to the zone of ℓ.

outline of the proof continued

We first assumed ℓ_1 intersects ℓ only at one point v, but the degree of the vertex v may be much higher, for example: u and/or w may collide with v.

Exercise 2: Examine the case u collides with v.
How many new edges appear in this case?

Exercise 3: Examine the case *u and w* collide with v.
How many new edges appear in this case?

summary and exercises

We closed chapter 8 in the textbook.

The zone theorem proves the $O(n^2)$ cost of an incremental algorithm to construct the subdivision defined by a set of n lines.

Consider the following activities, listed below.

1. Write the solutions to exercises 1, 2, and 3.
2. Consult the CGAL documentation and example code on arrangements of lines.
3. Consider the exercises 10, 12, 13 in the textbook.