The Zone Theorem

1. Arrangements of Lines
   - an incremental algorithm
   - the zone of a line
   - the zone theorem

2. Levels and Discrepancy
   - counting the number of lines
   - computing the level at vertices

3. Proof of the Zone Theorem
   - induction on the number of lines
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five lines in the plane induce a subdivision

The arrangement of lines is stored in a doubly connected edge list, within a bounding box.
an incremental algorithm

Algorithm \textsc{ConstructArrangement}(L)

Input: a set \( L \) of \( n \) lines.
Output: \( A(L) \), stored in doubly connected edge list, within a bounding box \( B(L) \).

1. compute \( B(L) \) enclosing all vertices of \( A(L) \)
2. construct a doubly connected edge list \( \mathcal{D} \) to store \( B(L) \)
3. for \( i \) from 1 to \( n \) do
4. find the edge \( e \) on \( B(L) \) that contains leftmost intersection point of \( \ell_i \) and \( A_{i-1} \)
5. let \( f \) be the bounded face incident to \( e \)
6. while \( f \) is not outside \( B(L) \) do
7. split \( f \), update \( \mathcal{D} \)
8. set \( f \) to the next face intersected by \( \ell_i \)
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the zone of a line – an example

The blue line intersects 5 of the 11 faces.
the zone of a line – definitions

Definition (the closure of a face in a subdivision)
Let $f$ be a face in a subdivision. The closure $\overline{f}$ of the face $f$ is $f$ and all its vertices and edges.

Definition (the zone of a line in an arrangement)
Let $A(L)$ be a line arrangement and $\ell$ be a line. The zone of the line $\ell$ in the arrangement $A(L)$ is

$$\{ f \text{ face of } A(L) \mid f \cap \ell \neq \emptyset \}.$$

Definition (the zone complexity)
Let $A(L)$ be a line arrangement and $\ell$ be a line. The zone complexity of $\ell$ in $A(L)$ is the sum of the number of vertices, the number of edges, and the number of faces in the zone of $\ell$ in $A(L)$.
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Theorem (the zone theorem)

Let $L$ be a set of $m$ lines and $\ell$ be some line. The zone complexity of a line $\ell$ in the arrangement $A(L)$ is $O(m)$.

By the zone theorem, the cost of the incremental algorithm is quadratic.

Theorem (cost of $\text{CONSTRUCT}\text{ARRANGEMENT}$)

A doubly connected edge list for the arrangement induced by a set of $n$ lines in the plane can be constructed in $O(n^2)$ time.
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the dual of the discrepancy problem

Given a line $\ell$, we want to count all points below $\ell$.

Given the point $\ell^*$, count the lines below $\ell^*$. 
the level of a point in an arrangement

Definition (level of a point in an arrangement)
Given an arrangement $A(L)$ of lines and a point $p$, the *level of the point $p$ in $A(L)$* is the number of lines strictly above $p$. 

![Diagram showing points and lines with their respective levels labeled 0, 1, 2, 3, and 4.](image-url)
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For any line \( \ell \) do the following:

1. compute the level at the leftmost vertex,
2. while not at the rightmost vertex on \( \ell \) do
3. walk to the next vertex \( v \) on \( \ell \) and compute the level of \( v \).
In an arrangement of \( n \) lines, and a given line \( \ell \), computing the level of the leftmost vertex on \( \ell \) runs in \( O(n) \).

→ for the vertex \( v \) on \( \ell \) with the smallest \( x \)-coordinate, check all other \( n - 1 \) lines to see whether \( v \) lies below.
computing the level of the next vertex

In an arrangement of $n$ lines, and a given line $\ell$, computing the level of the next vertex on $\ell$ also runs in $O(n)$.

→ in the walk from one vertex to the next, we follow the edges in the doubly connected edge list and update the level as follows:

$+1$ if the edge we follow goes down,

$-1$ if the edge we follow goes up.
computing the discrete measure in quadratic time

The discrete measure of $S$ in $U$ is $\mu_S(h) = \#(S \cap U)/\#S$.

The dual of the sample set $S$ of points is the set of lines $S^*$. We count the levels of the vertices in the arrangement $A(S^*)$.

**Theorem (cost of half plane discrepancy)**

The half plane discrepancy of a set $S$ of $n$ points in the unit square $U$ can be computed in $O(n^2)$ time.
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Theorem (the zone theorem)

Let $L$ be a set of $m$ lines and $\ell$ be some line. The zone complexity of a line $\ell$ in the arrangement $A(L)$ is $O(m)$.

Outline of the proof:

- Choose the coordinate system so that $\ell$ is the $x$-axis.
- Each edge in $A(L)$ bounds two faces.
  - An edge is a left bounding face for the face to its right.
  - An edge is a right bounding face for the face to its left.
- In the zone of $\ell$, the number of left bounding edges $\leq 5m$.

The theorem follows from the last statement.
The number of left bounding edges

Lemma (the number of left bounding edges)

Let \( L \) be a set of \( m \) lines and \( \ell \) be the x-axis. In the zone of \( \ell \) in \( A(L) \), the number of left bounding edges \( \leq 5m \).

The lemma is proven by induction on \( m \).

- **The base case:** \( m = 1 \), only one line in \( L \), 5 is indeed an upper bound to the number of left bounding edges.

- **The general case.**
  Let \( \ell_1 \) be the line in \( L \) that has the rightmost intersection with \( \ell \).

  We apply the induction hypothesis to \( A(L \setminus \{\ell_1\}) \):
  in \( A(L \setminus \{\ell_1\}) \), the number of left bounding edges \( \leq 5(m - 1) \).

  Need to show:
  no more than 5 new left bounding edges when \( \ell_1 \) is added.
the general case

We first assume \( \ell_1 \) intersects \( \ell \) only at one point \( v \):

We see 5 new edges.

1. The edge on \( \ell_1 \), spanned by \((u, v)\).
2. The edge on \( \ell_1 \), spanned by \((v, w)\).
3. The edge on \( \ell_2 \), starting at \( u \).
4. The edge on \( \ell \), starting at \( v \).
5. The edge on \( \ell_3 \), starting at \( w \).
We first assume $\ell_1$ intersects $\ell$ only at one point $v$:

The 5 new edges may not the only new edges. However, other new edges are above the vertex $u$ or below $w$ and therefore do not belong to the zone of $\ell$. 

\[ \ell \]
We first assumed \( \ell_1 \) intersects \( \ell \) only at one point \( v \), but the degree of the vertex \( v \) may be much higher, for example: \( u \) and/or \( w \) may collide with \( v \).

**Exercise 1:** Examine the case \( u \) collides with \( v \). How many new edges appear in this case?

**Exercise 2:** Examine the case \( u \) and \( w \) collide with \( v \). How many new edges appear in this case?
recommended assignments

We finished chapter 8 in the textbook.

Consider the following activities, listed below.

1. Write the solutions to exercises 1 and 2.
2. Consult the CGAL documentation and example code on arrangements of lines.
3. Consider the exercises 10, 12, 13 in the textbook.